



## Three-level models for continuous outcomes

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### 3.1.1 A 3-level random intercept model using class and school as IDs

In other models it has been shown that both school and classroom contribute to the explanation of the total variation of the POSTTHKS scores. We now construct a three-level model that uses both CLASS and SCHOOL as level-2 and level-3 IDs.

#### 3.1.1.1 The model

The level-1 and level-2 models are the same as the previous two models, as shown below.

Level-1 model ( $k = 1, \dots, n_{ij}$ )

$$\text{POSTTHKS}_{ijk} = b_{0ij} + e_{ijk},$$

$$e_{ijk} : NID(0, \sigma^2)$$

Level-2 model ( $j = 1, \dots, n_i$ )

$$b_{0ij} = b_{0i} + b_{1i} \text{CC}_{ij} + b_{2i} \text{TV}_{ij} + b_{3i} (\text{CC}_{ij} \times \text{TV}_{ij}) + v_{0ij}$$

$$v_{0ij} : NID(0, \sigma_{v(2)}^2)$$

Level-3 model ( $i = 1, \dots, N$ )

$$b_{0i} = \beta_0 + v_{0i}$$

$$b_{1i} = \beta_1$$

$$b_{2i} = \beta_2$$

$$b_{3i} = \beta_3$$

$$v_{0i} : NID(0, \sigma_{v(3)}^2)$$

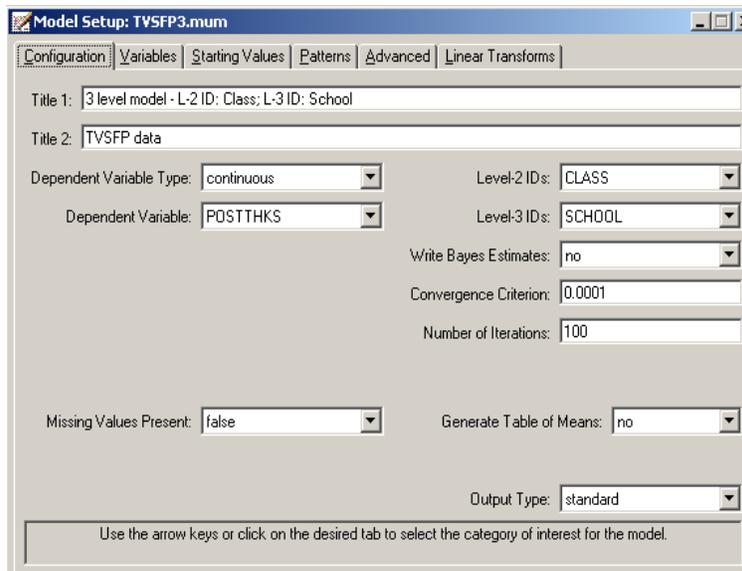
In this mixed model the intercept  $b_{0ij}$  is estimated by a level-2 equation. It indicates that classroom  $j$ 's initial value is not only determined by the population average  $b_{0i}$ , but also by the classroom difference  $v_{0ij}$ . The level-2-intercept  $b_{0ij}$  is estimated by a level-3 equation which takes the school difference  $v_{0i}$  into consideration, where  $i$  denotes the school ID.

The above model can also be written in the following format.

$$\text{POSTTHKS}_{ijk} = \beta_0 + \beta_1 \text{CC}_{ij} + \beta_2 \text{TV}_{ij} + \beta_3 (\text{CC}_{ij} \times \text{TV}_{ij}) + v_{0ij} + v_{0i} + e_{ijk}.$$

### 3.1.1.2 Setting up the analysis

We modify our model setup saved to the syntax file **TVSFP1.mum** by first using the **Open Existing Model Setup** option on the **File** menu of the **TVSFP.ss3** window to retrieve the syntax file. Then click on **File, Save as** to save the model setup in a new file, such as **TVSFP3.mum**.

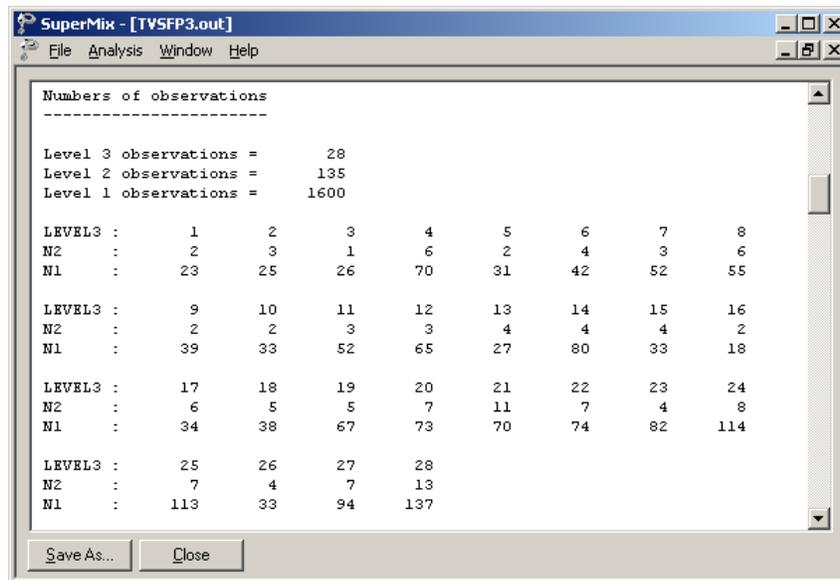


Next, we change the string in the **Title 1** text box on the **Configuration** screen, and select **SCHOOL** as the **Level-3 ID** as shown below. We now have both level-2 and level-3 IDs selected. Keep all the other settings unchanged. Save the changes to the file **TVSFP3.mum** and select the **Run** option on the **Analysis** menu to produce the output file **TVSFP3.out**.

### 3.1.1.3 Discussion of results

#### Model and data description

The **number of observations** section clearly shows the hierarchical structure of the data. The data contains 1600 students from 135 classes nested in 28 schools. In school number 20 (LEVEL 3: 20), the data of 73 students (N1: 73) from 7 (N2: 7) classes are present in this data set.



#### Fixed effects estimates

As shown below, the estimates are not markedly different from the estimates of the previous two models.

SuperMix - [TV5FP3.out]

File Analysis Window Help

TITLE1: 3 level model - L-2 ID: Class\_ L-3 ID: School

Maximum likelihood estimates

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Fixed regressor(s)

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Variable	Estimate	Std. Err.	Z-value	p-value
intcept	2.35537	0.12784	18.42442	0.00000
CC	0.61496	0.18243	3.37095	0.00075
TV	0.17156	0.17858	0.96068	0.33671
CCxTV	-0.35078	0.25465	-1.37750	0.16836

Save As... Close

Both the deviance and Akaike information criterion (AIC) are slightly higher than the previous model. The SBC is smaller.

SuperMix - [TV5FP3.out]

File Analysis Window Help

Log Likelihood	=	-2745.5166
-2 Log Likelihood (Deviance)	=	5491.0333
Akaike's Information Criterion	=	5505.0333
Schwarz's Bayesian Criterion	=	5514.3587
Number of free parameters	=	7

Save As... Close

### Random effect estimates

The estimates for the random part of the model are reported next.

SuperMix - [TV5FP3.out]

File Analysis Window Help

Variance/covariance components

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Level 3	Estimate	Std. Err.	Z-value	p-value
intcept /intcept	0.05660	0.03032	1.86663	0.06195
Level 2	Estimate	Std. Err.	Z-value	p-value
intcept /intcept	0.07903	0.03172	2.49176	0.01271
Level 1	Estimate	Std. Err.	Z-value	p-value
intcept /intcept	1.72652	0.06349	27.19236	0.00000

Save As... Close

The estimated level-2 random effect is highly significant ( $p = 0.08$ ), but the level-3 is not ( $p = 0.06$ ).

### 3.1.1.4 Interpreting the results

#### Fit statistics and % variation explained

The variation of POSTTHKS scores can be explained by individual differences, classroom differences and school differences.

For schools,

$$\frac{0.05660}{0.05660 + 0.07903 + 1.72652} \times 100\% = 3.04\%,$$

while for classrooms

$$\frac{0.07903}{0.05660 + 0.07903 + 1.72652} \times 100\% = 4.24\%.$$

As calculated above, the school difference contributes 3.04% to the explanation of the total variance in the outcome, and classroom difference contributes 4.24%. The rest, 92.72% of the variation, is explained by the student differences.

#### Comparison of models

In Table 3.6 the estimates of the previous three models and OLS in this chapter are summarized. The three-level estimates all lie between the corresponding two level-2 estimates.

**Table 3.6: Comparison of OLS and mixed model results**

	OLS Estimates	Mixed Model		
		L-2 model		L-3 model
		L-2 ID: CLASS	L-2 ID: SCHOOL	L-2 ID: CLASS L-3 ID: SCHOOL
intcept	2.361 (0.066)	2.341 (0.092)	2.361 (0.124)	2.355 (0.128)
CC	0.607 (0.096)	0.589 (0.133)	0.652 (0.178)	0.615 (0.182)
TV	0.177 (0.094)	0.120 (0.131)	0.198 (0.175)	0.172 (0.179)
CCxTV	-0.323 (0.137)	-0.247 (0.189)	-0.417 (0.250)	-0.351 (0.255)
Deviance		5498.168	5501.438	5491.033
AIC		5510.168	5513.438	5505.033
SBC		5527.600	5521.431	5514.359
Number of free parameters		6	6	7

### 3.1.2 A 3-level random intercept model including pre-THKS score

The PRETHKS variable indicates the observed score before implementation of intervention. It might have an impact on the POSTTHKS scores. In this section, a three-level model including the PRETHKS as predictor is fitted.

#### 3.1.2.1 The model

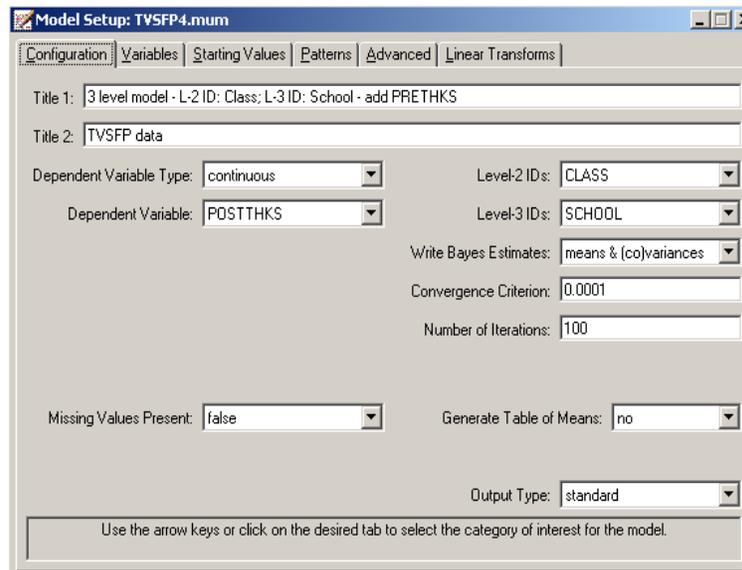
The only difference between this model and the previous one is the addition of the variable PRETHKS as a covariate:

$$\text{POSTHKS}_{ijk} = \beta_0 + \beta_1 \text{CC}_{ij} + \beta_2 \text{TV}_{ij} + \beta_3 (\text{CC}_{ij} \times \text{TV}_{ij}) + \beta_4 (\text{PRETHKS}_{ijk}) + v_{0ij} + v_{0i} + e_{ijk}.$$

From the subscripts associated with the coefficients, we note that while CC, TV and CCxTV were measured at a classroom level, the pre-intervention score PRETHKS is measured on the individual level. Such a variable may also be referred to as a level-1 predictor, while CC, TV and CCxTV may be called level-2 predictors, covariates, or mediating effects.

### 3.1.2.2 Setting up the analysis

The easiest way to set up this model is to modify the model setup in the syntax file **TVSFP3.mum** by first using the **Open Existing Model Setup** option on the **File** menu. Then click on **File, Save as** to save the model setup in a new file, such as **TVSFP4.mum**.



Next, we change the string in the **Title 1** text box on the **Configuration** screen. Notice that we would like to request Bayes estimates as part of the program output. To do so, select **means & (co)variances** option from the **Write Bayes Estimates** drop down list as shown above.

Click on the **Variables** tab and select PRETHKS as an additional **Explanatory Variable** by checking the corresponding **E** check box. Save the changes to the file **TVSFP4.mum** and select the **Run** option on the **Analysis** menu to produce the output file **TVSFP4.out**.

### 3.1.2.3 Discussion of results

#### Fixed effects estimates and descriptive statistics

As shown below, the estimated coefficient of PRETHKS is highly significant. The estimate of the intercept coefficient decreased because part of the variation in the intercept can now be explained by PRETHKS.

SuperMix - [TVSFP4.out]

File Analysis Window Help

TITLE1: 3 level model - L-2 ID: Class\_ L-3 ID: School - add PRETHKS

Maximum likelihood estimates

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Fixed regressor(s)

-----

Variable	Estimate	Std. Err.	Z-value	p-value
intcept	1.69700	0.11666	14.54688	0.00000
CC	0.63919	0.14722	4.34189	0.00001
TV	0.17811	0.14365	1.23986	0.21503
CCxTV	-0.32042	0.20551	-1.55910	0.11897
PRETHKS	0.30720	0.02584	11.88788	0.00000

Save As... Close

The fit statistics are given below. A comparison of these two three-level examples will be given in the next section.

SuperMix - [TVSFP4.out]

File Analysis Window Help

Log Likelihood	=	-2678.6793
-2 Log Likelihood (Deviance)	=	5357.3586
Akaike's Information Criterion	=	5373.3586
Schwarz's Bayesian Criterion	=	5384.0163
Number of free parameters	=	8

Save As... Close

### Random effect estimates

The third-level random intercept estimate is not significant at a 5% level of significance, which implies that after taking PRETHKS into account, the school differences are not significant.

SuperMix - [TVSFP4.out]

File Analysis Window Help

Variance/covariance components

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Level	Estimate	Std. Err.	Z-value	p-value
Level 3				
intcept /intcept	0.02575	0.01971	1.30665	0.19133
Level 2				
intcept /intcept	0.06358	0.02767	2.29768	0.02158
Level 1				
intcept /intcept	1.60201	0.05889	27.20544	0.00000

Save As... Close

### 3.1.2.4 Interpreting the results

#### Estimated outcomes for different groups

For example, if a typical student who only participated in television intervention had a PRETHKS score of 2 (CC = 0; TV = 1; CCxTV = 0), the expected POSTTHKS score is calculated as follows:

$$\begin{aligned}\widehat{\text{POSTTHKS}}_{ijk} &= \hat{\beta}_{00} + \hat{\beta}_{02} \text{TV}_{ij} + \hat{\beta}_{04} (\text{PRETHKS}_{ijk}) \\ &= 1.697 + 0.17811 + 2 \times 0.3072 \\ &= 2.48951.\end{aligned}$$

#### ICCs and R square

##### ICCs

The so-called ICC (interclass correlation) measures the proportion of variation in the outcome variable between units at the different levels. It is occasionally referred to as the cluster effect, and is defined as the ratio of the between-cluster variance to the total variance. From the output for the random effects, we have

$$\text{Level-1: estimated (error var)} = 1.6020$$

$$\text{Level-2: estimated (class var)} = 0.0636$$

$$\text{Level-3: estimated (school var)} = 0.0258.$$

Based on this information, we can calculate the ICCs as shown below.

Similarity of students within the same school:

$$ICC = \frac{0.0258}{1.6020 + 0.0636 + 0.0258} = 0.0153$$

Similarity of students within the same classrooms (and schools):

$$ICC = \frac{0.0636 + 0.0258}{1.6020 + 0.0636 + 0.0258} = 0.0529$$

Similarity of classes within the same school:

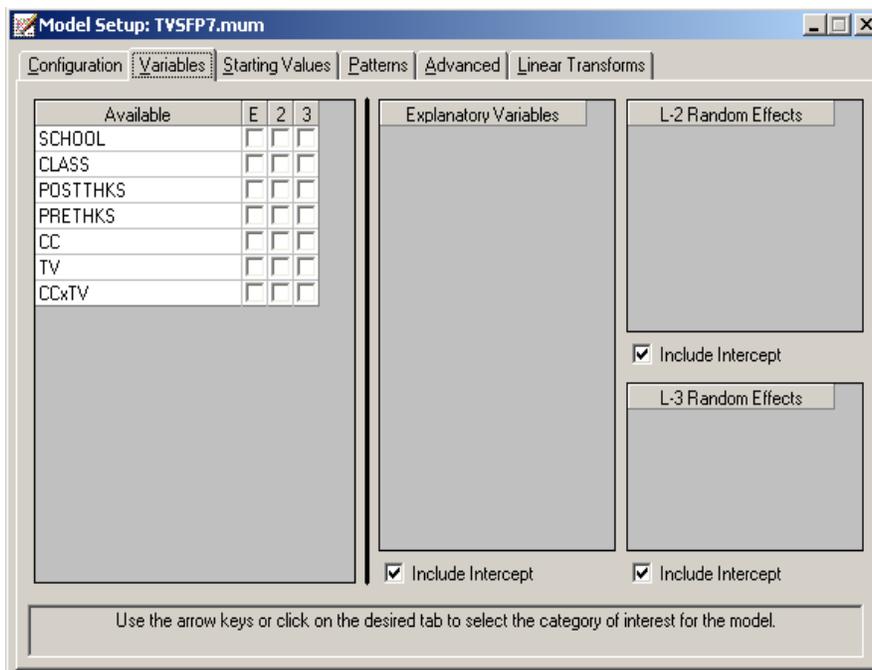
$$ICC = \frac{0.0258}{0.0636 + 0.0258} = 0.289$$

##### R-square

Another way to evaluate the explanation of variation in the outcome is to compute a statistic analogous to the familiar  $R^2$  used in multiple linear regression. In a multilevel model,

however, there is an  $R^2$  for each variance component. Use of these statistics is not without problems, however, because the  $R^2$  may at times have negative values, and in other cases the addition of explanatory variables can lead to an increase rather than a decrease in variance components. The more complex a hierarchical model is, the more likely is the occurrence of anomalies in variance-explained statistics.

To calculate the  $R^2$ s for different levels of the level-3 model, we first need to get the variances for the null model, which is a 3-level model with no covariates. Open the file **TVSFP4.mum**, click on the **Variables** tab, and uncheck the check boxes of the selected **Explanatory Variables** as shown below.



Save the setup as **TVSFP7.mum** and run the model to get the following output of the variance/covariance component.

The  $R^2$  s are calculated as

$$R_1^2 = 1 - \frac{\hat{\sigma}_p^2}{\hat{\sigma}_0^2} \quad R_2^2 = 1 - \frac{\hat{\sigma}_{v(2)p}^2}{\hat{\sigma}_{v(2)0}^2} \quad R_3^2 = 1 - \frac{\hat{\sigma}_{v(3)p}^2}{\hat{\sigma}_{v(3)0}^2}$$

where subscript 0 refers to a model with no covariates (*i.e.*, the null model, **TVSFP7.out**) and subscript  $p$  refers to a model with  $p$  covariates (*i.e.*, the full model, **TVSFP4.out**). The  $R^2$  s for different levels are given in Table 3.7.

Variance/covariance components				
Level	Estimate	Std. Err.	Z-value	p-value
Level 3				
intcept /intcept	0.11032	0.04573	2.41251	0.01584
Level 2				
intcept /intcept	0.08481	0.03281	2.58504	0.00974
Level 1				
intcept /intcept	1.72367	0.06341	27.18391	0.00000

**Table 3.7:  $R^2$  values for a set of nested models**

level	variance	null	full	$R^2$
1 (students)	$\hat{\sigma}^2$	1.724	1.602	.071
2 (classrooms)	$\hat{\sigma}_{v(2)}^2$	.085	.064	.247
3 (schools)	$\hat{\sigma}_{v(3)}^2$	.110	.026	.764

In the current example, only the intercept coefficient is allowed to vary randomly over classrooms and schools, thus making the calculation of the  $R^2$  relatively straightforward. In the case of models with random slopes, the calculation of  $R^2$  statistics becomes more difficult. For an extensive discussion of the rationale and calculation of  $R^2$  statistics, the user is referred to Snijders & Bosker (2000).

### Model fit statistics and comparison of models

Now, we consider two level-2 models using the same covariates but different level-2 IDs: one uses CLASS as level-2 ID, the other uses SCHOOL. The models' setups are given in **TVSFP5.mum** and **TVSFP6.mum**. The comparison of estimates is summarized in the Table 3.8.

**Table 3.8: Comparison of OLS and mixed model results**

	OLS Estimates	Mixed Model		
		L-2 model		L-3 model
		L-2 ID: CLASS	L-2 ID: SCHOOL	L-2 ID: CLASS L-3 ID: SCHOOL
intcept	1.6613 (0.0844)	1.6776 (0.0988)	1.6952 (0.1145)	1.6970 (0.1167)
CC	0.6406 (0.0921)	0.6330 (0.1186)	0.6601 (0.1440)	0.6392 (0.1472)
TV	0.1987 (0.0900)	0.1597 (0.1167)	0.2024 (0.1401)	0.1781 (0.1437)
CCxTV	-0.3216 (0.1303)	-0.2747 (0.1678)	-0.3697 (0.2011)	-0.3204 (0.2055)
PRETHKS	0.3252 (0.0259)	0.3116 (0.0258)	0.3103 (0.0259)	0.3072 (0.0258)
error variance		1.6030 (0.0589)	1.6523 (0.0589)	1.6020 (0.0589)
class variance		0.0870 (0.0277)		0.0636 (0.0277)
school variance			0.0372 (0.0184)	0.0258 (0.0197)
Deviance		5359.9641	5366.0133	5357.3586
AIC		5373.9641	5380.0133	5373.3586
SBC		5394.3010	5389.3387	5384.0163
Number of free parameters		7	7	8

When comparing the deviances, AIC and SBC of the level-3 model with the level-2 models, we conclude that the three-level model has a better fit to the data.

### 3.1.2.5 Residuals

#### Level-2 Bayes results

Returning to the **TVSFP4.mum** output, click on the **Analysis** menu of the output window or the model set up window, and note that **View Level-2 Bayes Results** is now activated. Select the option to open the level-2 Bayes results.



Note that the default extension for the level-2 Bayes estimates is **.ba2**. Part of the file is shown below.

Level-3 ID	Level-2 ID	Observations	Empirical Bayes Coefficients	Empirical Bayes Estimate	Estimated Variance	Coefficient Name
401.00	401101.00	18	1	0.13415	0.33783E-01	intcept
401.00	401102.00	21	1	0.40196E-01	0.30746E-01	intcept
402.00	402101.00	17	1	0.91264E-01	0.34802E-01	intcept
402.00	402102.00	16	1	-0.78120E-01	0.35947E-01	intcept
405.00	405101.00	15	1	0.11404	0.37454E-01	intcept
405.00	405102.00	16	1	-0.23619	0.36287E-01	intcept
405.00	405103.00	21	1	0.58924E-01	0.31114E-01	intcept
407.00	407101.00	21	1	0.38397	0.31261E-01	intcept
407.00	407102.00	27	1	0.19960	0.26267E-01	intcept
407.00	407103.00	17	1	-0.20588	0.35282E-01	intcept
408.00	408101.00	3	1	-0.19175	0.56599E-01	intcept
408.00	408102.00	2	1	-0.17972	0.58804E-01	intcept
408.00	408103.00	15	1	0.23679	0.37123E-01	intcept
408.00	408104.00	7	1	0.20771E-01	0.48833E-01	intcept

The representations of these seven columns are given in order below:

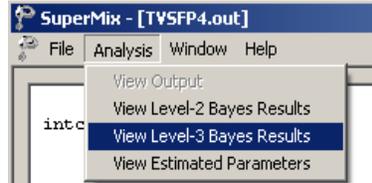
- Column 1: level-3 ID, which is school in our example.
- Column 2: level-2 ID, which refers to classroom.
- Column 3: number of the observations within level-2 ID, number of students within each classroom.
- Column 4: the number of the empirical Bayes coefficients.
- Column 5: the empirical Bayes estimate.
- Column 6: the estimated variance of the Bayes coefficient.
- Column 7: the name of the associated coefficient as used in the model.

Classroom 407102 has the largest Bayes estimate with a value of 0.38397. When considering the class difference, the predicted POSTTHKS score for a student in this specific class who only participated in television intervention with a PRETHKS score of 2 (CC = 0; TV = 1; CCxTV = 0) is calculated as follows.

$$\begin{aligned}
 \widehat{POSTTHKS}_{ijk} &= \widehat{\beta}_0 + \widehat{\beta}_2 TV_{ij} + \widehat{\beta}_4 (\widehat{PRETHKS}_{ijk}) + \widehat{u}_{0i} \\
 &= 1.697 + 0.17811 + 2 \times 0.3072 + 0.38397 \\
 &= 2.87348.
 \end{aligned}$$

### Level-3 Bayes results

Similarly, the level-3 Bayes results can be viewed by clicking on the **Analysis, View Level-3 Bayes Results**.



Part of the **TVSFP.ba3** is shown below.

The image shows a window titled 'SuperMix - [TVSFP4.ba3]'. The main area displays a table of estimated parameters for 15 classrooms. The table has five columns: Classroom ID, a binary variable (likely TV), two intercept estimates in scientific notation, and the label 'intcept'. The value 0.15296 is highlighted in the third column for classroom 407.

Classroom ID	TV	Estimate 1	Estimate 2	Label
403.00	1	0.58013E-01	0.21068E-01	intcept
404.00	1	0.24995E-01	0.19858E-01	intcept
193.00	1	0.24110E-01	0.21358E-01	intcept
194.00	1	0.45842E-01	0.14584E-01	intcept
196.00	1	0.29206E-01	0.19821E-01	intcept
197.00	1	0.46312E-01	0.18102E-01	intcept
198.00	1	-0.93556E-01	0.17253E-01	intcept
199.00	1	-0.83225E-01	0.16132E-01	intcept
401.00	1	0.70605E-01	0.19034E-01	intcept
402.00	1	0.53231E-02	0.19502E-01	intcept
405.00	1	-0.25606E-01	0.17253E-01	intcept
407.00	1	0.15296	0.16541E-01	intcept
408.00	1	-0.46129E-01	0.19626E-01	intcept
409.00	1	0.47700E-01	0.15045E-01	intcept

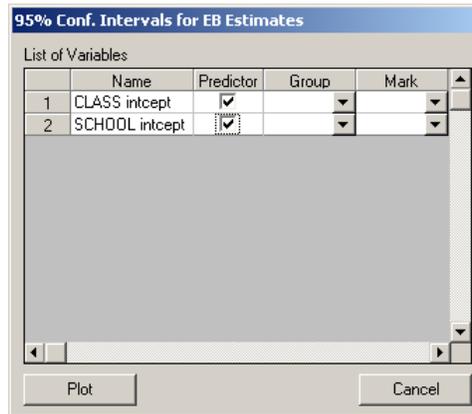
The same classroom (ID = 407102) discussed above is nested in school number 407. Now, considering both the class and school differences, the estimated POSTTHKS for a student from this classroom who only participated in television intervention with a pre-intervention score of 2 (CC = 0; TV = 1; CCxTV = 0) is calculated as follows.

$$\begin{aligned}
 \widehat{POSTTHKS}_{ijk} &= \hat{\beta}_0 + \hat{\beta}_2 TV_{ij} + \hat{\beta}_4 (\widehat{PRETHKS}_{ijk}) + \hat{v}_{0ij} + \hat{v}_{0i} \\
 &= 1.697 + 0.17811 + 2 \times 0.3072 + 0.38397 + 0.15296 \\
 &= 3.02644.
 \end{aligned}$$

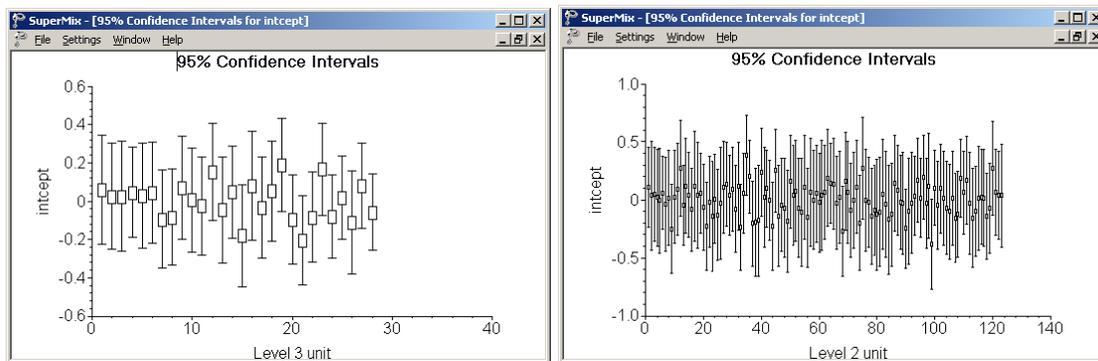
### Confidence intervals for random coefficients

The **Confidence Intervals** option on the **File, Model-based Graphs** menu provides the option to display confidence intervals for the empirical Bayes estimates of the random effects specified in a given model. This option is now used to examine the confidence intervals of the random intercepts for the schools, which represent the highest level of the hierarchy in the current example.

Select the **Confidence Intervals** option on the **File, Model-based Graphs** menu to activate the **95% Conf. Intervals for EB estimates** dialog box. Two graphs of the confidence intervals for the empirical Bayes estimates of the intercepts at the classroom level and school level are obtained by selecting CLASS intcept and SCHOOL intcept in the **Predictor** column before clicking **Plot**.



The graph obtained, as shown below, shows that, in general, the range of the confidence intervals for the level-3 empirical Bayes estimates of the intercepts is  $(-0.2; 0.2)$ , and the range for level-2 is about  $(-0.4; 0.4)$ .



**Figure 3.13: 95% confidence intervals for level-2 Bayes estimates**

The deviations from the estimated population intercept over schools are also apparent. Each confidence interval is obtained using the formula

$$\text{Empirical Bayes residual} \pm 1.96\sqrt{\text{var}(\text{Empirical Bayes residual})}.$$