Three-level models (HLM3)

Conventional Representation: Level 1

At level 1, we have

$$Y_{jk} = A_{jk}^{R}\pi_{jk}^{R} + A_{jk}^{F}\pi_{jk}^{F} + e_{jk} . \qquad (1)$$

Here

- Y_{jk} is the n_{jk} by 1 outcome vector,
- A_{jk}^{R} is the n_{jk} by P^{R} matrix of predictors having random effects at level 2;
- π_{jk}^{R} is the P^{R} vector of level-1 coefficients that are random at level 2;
- A_{jk}^{F} is the n_{jk} by P^{F} matrix of predictors having fixed effects at level 2; and
- π_{jk}^{F} is the P^{F} vector of level-1 coefficients having fixed effects at level 2.

Conventional Representation: Level 2

At level 2 we have

$$\begin{aligned} \pi_{jk}^{R} &= X_{jk}^{RR}\beta_{k}^{RR} + X_{jk}^{RF}\beta_{k}^{RF} + r_{jk} \\ \pi_{jk}^{F} &= X_{jk}^{FR}\beta_{k}^{FR} + X_{jk}^{FF}\beta_{k}^{FF} \end{aligned} \qquad (2)$$

- X_{jk}^{RR} is a P^{R} by Q^{RR} matrix of predictors of π_{jk}^{R} having random effects at level 3;
- β_{k}^{RR} is the Q^{RR} by 1 vector of level-2 coefficients associated with X_{jk}^{RR};
- r_{jk} is a P^{R} by 1 vector of level-2 random effects;
- X_{jk}^{RF} is the P^{R} by Q^{RF} matrix of predictors of π_{jk}^{R} having fixed effects at level 3;
- β_{k}^{RF} is the Q^{RF} by 1 vector of level-2 coefficients associated with X_{jk}^{RF};
- X_{jk}^{FR} is the P^{F} by Q^{FR} matrix of predictors of π_{jk}^{F} having fixed effects at level 3;

- β_k^{FR} is the Q^{FR} by 1 vector of level-2 coefficients associated with X_{jk}^{FR};
- X_{jk}^{FF} is the P^F by Q^{FF} matrix of predictors of π_{jk}^F having fixed effects at level 3;
- β_k^{FF} is the Q^{FF} by 1 vector of level-2 coefficients associated with X_{jk}^{FF}.

Conventional Representation, Level 3

At level 3 we have

$$\begin{aligned}\beta_k^{RR} &= W_k^{RR}\gamma^{RR} + u_k^{RR}\\ \beta_k^{RF} &= W_k^{RF}\gamma^{RF}\\ \beta_k^{FR} &= W_k^{FR}\gamma^{FR} + u_k^{FR}\\ \beta_k^{FF} &= W_k^{FF}\gamma^{FF}\end{aligned} \quad (3)$$

Here

- W_k^{RR} is a Q^{RR} by S^{RR} matrix of predictors;
- γ^{RR} is a S^{RR} by 1 vector of fixed effects;
- u_k^{RR} is a Q^{RR} by 1 vector of random effects;
- W_k^{RF} is a Q^{RF} by S^{RF} matrix of predictors;
- γ^{RF} is an S^{RF} by 1 vector of fixed effects;
- W_k^{FR} is a Q^{FR} by S^{FR} matrix of predictors;
- γ^{FR} is a S^{FR} by 1 vector of fixed effects;
- u_k^{FR} is a Q^{FR} by 1 vector of random effects;
- W^{FF} is a Q^{FF} by S^{FF} matrix of predictors;
- γ^{FF} is a S^{FF} by 1 vector of fixed effects;

Revised Representation

The logic of allocating degrees of freedom is easy to follow if we represent the model as follows:

$$Y_{jk} = A_{jk}^R\left[X_{jk}^{RR}\left(W_k^{RR}\gamma^{RR} + u_k^{RR}\right) + X_{jk}^{RF}W_k^{RF}\gamma^{RF} + r_{jk}\right] + A_{jk}^F\left[X_{jk}^{FR}\left(W_k^{FR}\gamma^{FR} + u_k^{FR}\right) + X_{jk}^{FF}W_k^{FF}\gamma^{FF}\right] + e_{jk}$$
(4)

- The elements of π_{jk}^R are $\pi_{p^R jk}^R$, for p^R =1,…, P^R. The elements of π_{jk}^F are $\pi_{p^F jk}^F$, for p^F =1,…, P^F.
- The elements of β_k^{RR} are $\beta_{p^R q_{p^R}^R}$ for $q_{p^R}^R = 1,...,Q_{p^R}^R$. Define $Q^{RR} = \sum_{p^R=1}^{P^R} Q_{p^R}^R$, the number of rows of W_k^{RR}

- The elements of β_k^{RF} are $\beta_{p^R q^F_{p^R}}$ for $q^F_{p^R} = 1,...,Q^F_{p^R}$. Define $Q^{RF} = \sum_{p^R=1}^{P^R} Q^F_{p^R}$, the number of rows of W_k^{RF}.
- The elements of β_k^{FR} are $\beta_{p^F q^R_{p^F}}$ for $q^R_{p^F} = 1,...,Q^R_{p^F}$. Define $Q^{FR} = \sum_{p^F=1}^{P^F} Q^R_{p^F}$, the number of rows of W_k^{FR}.
- The elements of β_k^{FF} are $\beta_{p^F q^F_{p^F}}$ for $q^F_{p^F} = 1,...,Q^F_{p^F}$. Define $Q^{FF} = \sum_{p^F=1}^{P^F} Q^F_{p^F}$, the number of rows of W_k^{FF}.
- The elements of γ^{RR} are $\gamma^{RR}_{p^R q^R_{p^R} s_{p^R q^R_{p^R}}}$ for $s_{p^R q^R_{p^R}} = 1,...,S_{p^R q^R_{p^R}}$ where $\sum_{q^R_{p^R}=1}^{Q^R_{p^R}} S_{p^R q^R_{p^R}} = S^R_{p^R}$ and $\sum_{p^R=1}^{P^R} S^R_{p^R} = S^{RR}$ is the dimension of γ^{RR}.
- The elements of γ^{RF} are $\gamma^{RF}_{p^R q^F_{p^R} s_{p^R q^F_{p^R}}}$ for $s_{p^R q^R_{p^R}} = 1,...,S_{p^R q^F_{p^R}}$ where $\sum_{q^F_{p^R}=1}^{Q^F_{p^R}} S_{p^R q^F_{p^R}} = S^F_{p^R}$ and $\sum_{p^R=1}^{P^R} S^F_{p^R} = S^{RF}$ is the dimension of γ^{RF}.
- The elements of γ^{FR} are $\gamma^{FR}_{p^F q^R_{p^F} s_{p^F q^R_{p^F}}}$ for $s_{p^F q^R_{p^F}} = 1,...,S_{p^F q^R_{p^F}}$ where $\sum_{q^R_{p^F}=1}^{Q^R_{p^F}} S_{p^F q^R_{p^F}} = S^R_{p^F}$ and $\sum_{p^F=1}^{P^F} S^R_{p^F} = S^{FR}$ is the dimension of γ^{FR}.
- The elements of γ^{FF} are $\gamma^{FF}_{p^F q^F_{p^F} s_{p^F q^F_{p^F}}}$ for $s_{p^F q^R_{p^F}} = 1,...,S_{p^F q^F_{p^F}}$ where $\sum_{q^F_{p^F}=1}^{Q^F_{p^F}} S_{p^F q^F_{p^F}} = S^F_{p^F}$ and $\sum_{p^F=1}^{P^F} S^F_{p^F} = S^{FF}$ is the dimension of γ^{FF}.

Degrees of Freedom

1. For any element $\gamma^{RR}_{p^R q^R_{p^R} s_{p^R q^R_{p^R}}}$ of γ^{RR}, the degrees of freedom are $K - S_{p^R q^R_{p^R}}$.
2. For any element $\gamma^{RF}_{p^R q^F_{p^R} s_{p^R q^F_{p^R}}}$ of γ^{RF}, the degrees of freedom are $J - KQ^{RR} - S^{RF}$.
3. For any element $\gamma^{FR}_{p^F q^R_{p^F} s_{p^F q^R_{p^F}}}$ of γ^{FR}, the degrees of freedom are $K - S_{p^F q^R_{p^F}}$.

4. For any element $\gamma^{FF}_{p^F q^F_{p^F} s_{p^F q^F_{p^F}}}$ of γ^{FF}, the degrees of freedom are

$$N - K(Q^{RR} + Q^{FR}) - JP^{R} - S^{FF}.$$