## Executing analyses based on the MDM file

The steps involved are similar to the ones for HLM2. It is necessary to specify

1. the level-1 model,
2. the level-2 structural model, and
3. the level-1 coefficients as random or non-random.

Under HMLM, level-1 predictors having random effects must have the same value for all participants at a given occasion. If the user specifies a predictor not fulfilling this condition to have a random effect, such coefficients will be automatically set as non-random by the program. Furthermore, an extra step for selecting the covariance structure for the models to be estimated is needed. Figure 10.3 displays the model specified for our example. Figure 4 shows the dialog box where the covariance structure is selected.

## An annotated example of HMLM

In the example below (see NYS1.MLM) we specify AGE13 and AGE13S as predictors at level 1. At level 2, the model is unconditional. This is displayed in Fig. 3. We shall compare three alternative covariance structures:

- an unrestricted model,
- the homogeneous model, $\sigma_t^2 = \sigma^2$ for all $t$, and
- the heterogeneous model, which allows $\sigma_t^2$ to vary over time.

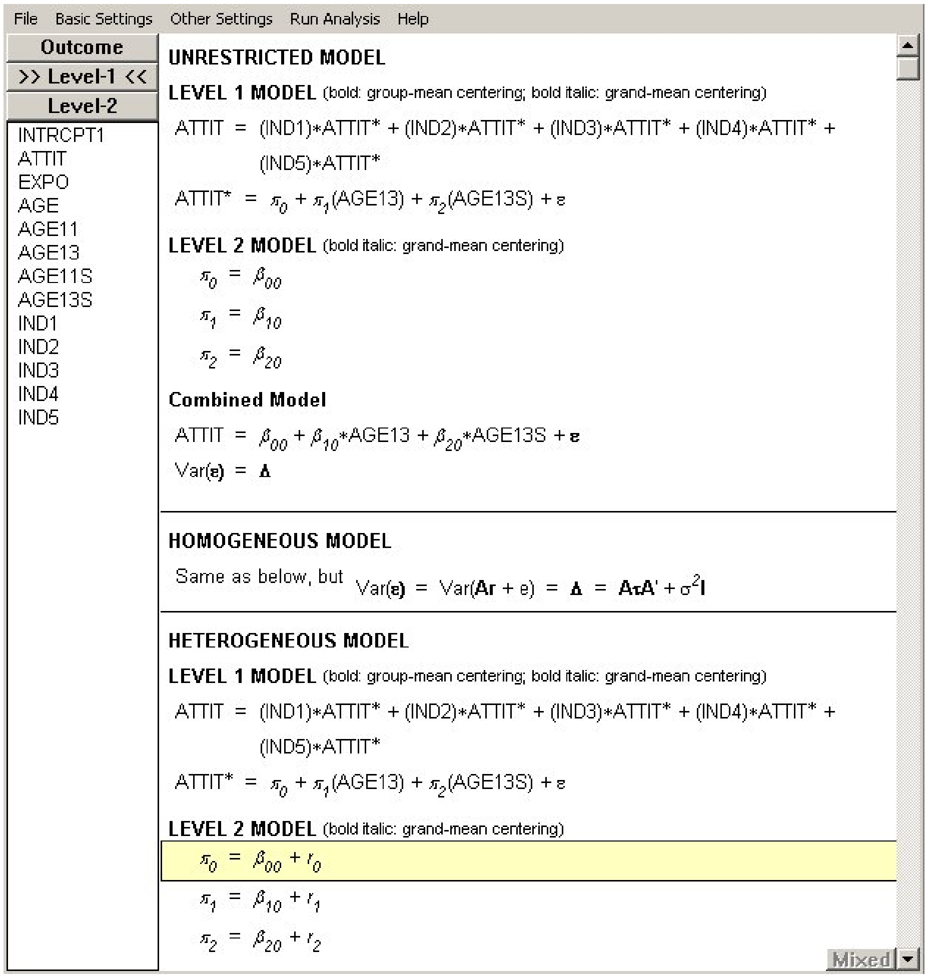


**Figure 3 Model specification window for the NYS example**

These three models are requested simply by checking the **Heterogeneous** option in the **Basic Model Specifications – HMLM** dialog box, as shown in Fig. 4.

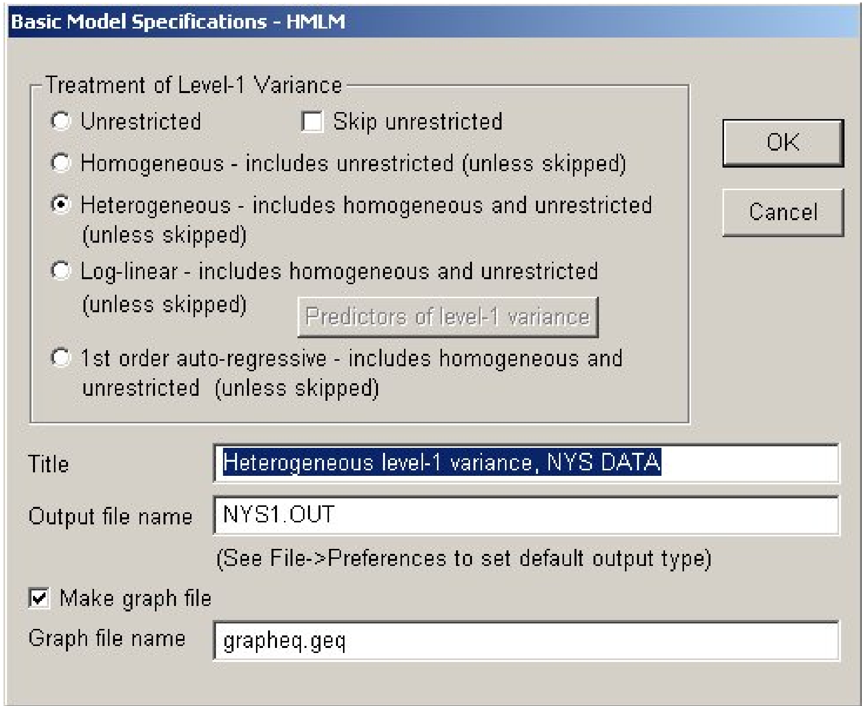


**Figure 4 Basic Model Specifications - HMLM dialog box**

Similarly, checking the **Log-linear** button will produce output on:

- the unrestricted model,
- the homogeneous model, and
- the log-linear model for $\sigma_t^2$.

In this case a modified model will be displayed, as shown in Fig. 5. To obtain this model, the **Predictors of level-1 variance** dialog box was used to select the variable EXPO.

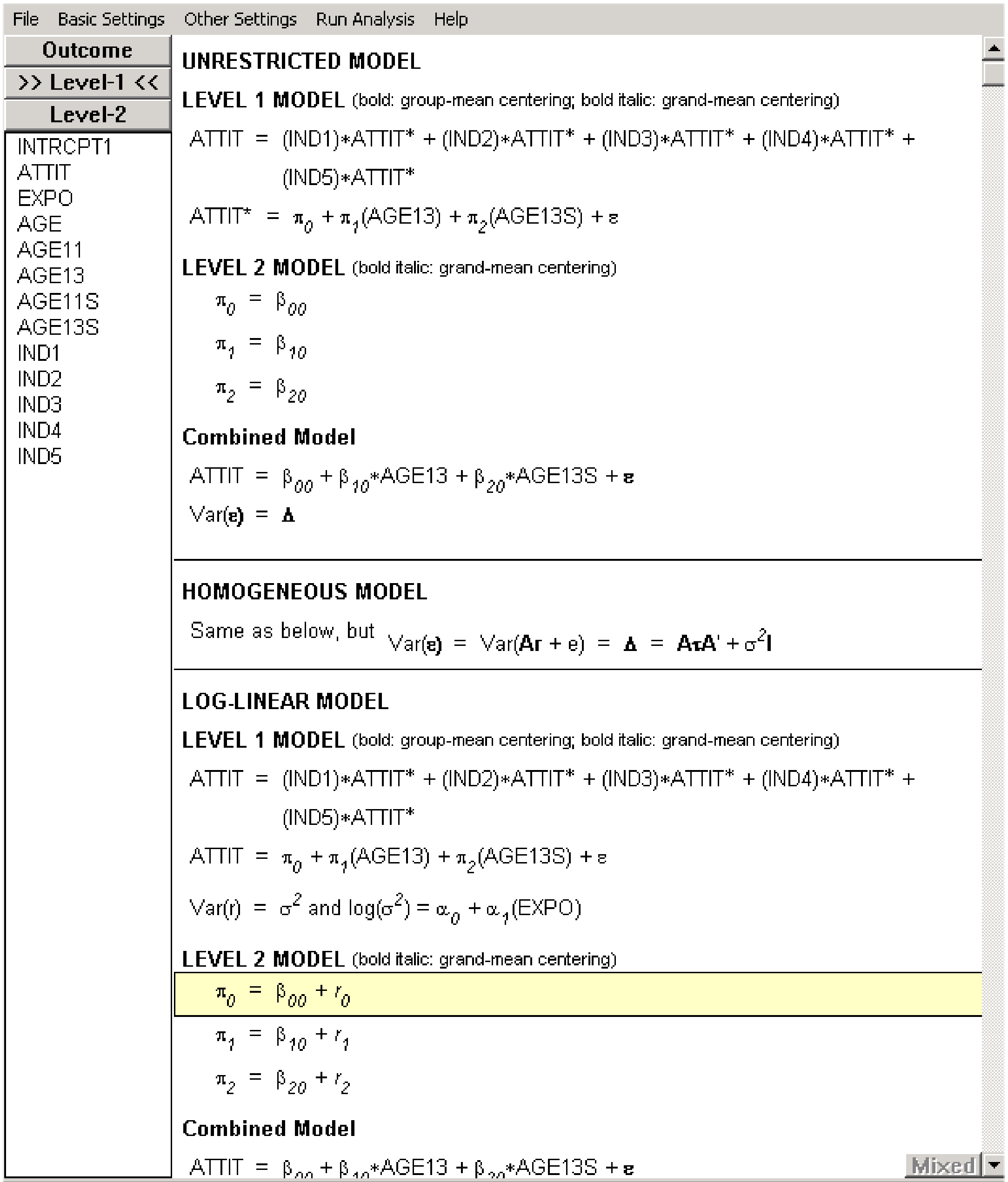


**Figure 5** **Model specification window for the NYS example: loglinear model selection**

And, again similarly, choosing the **1st order auto-regressive** option will produce unrestricted and homogeneous results in addition to first-order auto-regressive results.

The data source for this run = NYS.MDM
The command file for this run = nys1.hlm
Output file name = nys1.html
The maximum number of level-1 units = 1079
The maximum number of level-2 units = 239
The maximum number of iterations = 100

The outcome variable is ATTIT

The model specified for the fixed effects was:

| **Level-1 Coefficients** | Level-2 Predictors |
|---|---|
| INTRCPT1, $\pi_0$ | INTRCPT2, $\beta_{00}$ |
| # AGE13 slope, $\pi_1$ | INTRCPT2, $\beta_{10}$ |
| # AGE13S slope, $\pi_2$ | INTRCPT2, $\beta_{20}$ |

'#' - The residual parameter variance for this level-1 coefficient has been set to zero.

**Output for the Unrestricted Model**

**Summary of the model specified**

**Level-1 Model**

$$\text{ATTIT}_{mi} = (\text{IND1}_{mi})*\text{ATTIT}_{1i}^{*} + (\text{IND2}_{mi})*\text{ATTIT}_{2i}^{*} + (\text{IND3}_{mi})*\text{ATTIT}_{3i}^{*} + (\text{IND4}_{mi})*\text{ATTIT}_{4i}^{*} + (\text{IND5}_{mi})*\text{ATTIT}_{5i}^{*}$$

The level-1 model relates the observed data, $Y$, to the complete data, $Y^*$.

$$\text{ATTIT}_{ti}^{*} = \pi_{0i} + \pi_{1i}*(\text{AGE13}_{ti}) + \pi_{2i}*(\text{AGE13S}_{ti}) + \varepsilon_{ti}$$

**Level-2 Model**

$$\pi_{0i} = \beta_{00}$$
$$\pi_{1i} = \beta_{10}$$
$$\pi_{2i} = \beta_{20}$$

For the restricted model, there is no random variation between persons in regression coefficient $\beta_0$, $\beta_1$, and $\beta_2$ because all random variation has been absorbed into $\mathbf{\Delta}$.

$\text{Var}(\varepsilon_i) = \mathbf{\Delta}$

$\mathbf{\Delta}_{(0)}$

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 0.03507 | 0.01671 | 0.01889 | 0.02149 | 0.02486 |
| IND2 | 0.01671 | 0.04458 | 0.02779 | 0.02468 | 0.02714 |
| IND3 | 0.01889 | 0.02779 | 0.07272 | 0.05303 | 0.04801 |
| IND4 | 0.02149 | 0.02468 | 0.05303 | 0.08574 | 0.06636 |
| IND5 | 0.02486 | 0.02714 | 0.04801 | 0.06636 | 0.08985 |

The $5 \times 5$ matrix $\mathbf{\Delta}$ contains the maximum likelihood estimates of the five variances (one for each time point) and ten covariances (one for each pair of time points). The associated correlation matrix is printed below.

**Standard errors of $\boldsymbol{\Delta}$**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 0.00347 | 0.00304 | 0.00375 | 0.00413 | 0.00429 |
| IND2 | 0.00304 | 0.00434 | 0.00430 | 0.00457 | 0.00473 |
| IND3 | 0.00375 | 0.00430 | 0.00678 | 0.00631 | 0.00625 |
| IND4 | 0.00413 | 0.00457 | 0.00631 | 0.00811 | 0.00736 |
| IND5 | 0.00429 | 0.00473 | 0.00625 | 0.00736 | 0.00853 |

**$\boldsymbol{\Delta}$ (as correlations)**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 1.000 | 0.423 | 0.374 | 0.392 | 0.443 |
| IND2 | 0.423 | 1.000 | 0.488 | 0.399 | 0.429 |
| IND3 | 0.374 | 0.488 | 1.000 | 0.672 | 0.594 |
| IND4 | 0.392 | 0.399 | 0.672 | 1.000 | 0.756 |
| IND5 | 0.443 | 0.429 | 0.594 | 0.756 | 1.000 |

The $5 \times 5$ matrix above contains estimated standard errors for each element of $\boldsymbol{\Delta}$.

The value of the log-likelihood function at iteration 8 = 1.891335E+002

**Final estimation of fixed effects:**

| Fixed Effect | Coefficient | Standard error | *t*-ratio | Approx. *d.f.* | *p*-value |
|---|---|---|---|---|---|
| For INTRCPT1, $\pi_0$ | | | | | |
| INTRCPT2, $\beta_{00}$ | 0.320244 | 0.014981 | 21.377 | 238 | <0.001 |
| For AGE13 slope, $\pi_1$ | | | | | |
| INTRCPT2, $\beta_{10}$ | 0.059335 | 0.004710 | 12.598 | 238 | <0.001 |
| For AGE13S slope, $\pi_2$ | | | | | |
| INTRCPT2, $\beta_{20}$ | 0.000330 | 0.003146 | 0.105 | 238 | 0.917 |

The expected log attitude at age 13 is 0.320244. The mean linear growth rate of increase is estimated to be 0.059335, $t = 12.598$, indicating a highly significantly positive average rate of increase in deviant attitude at age 13. The quadratic rate is not statistically significant.

**Statistics for the current model**

Deviance = -378.266936
Number of estimated parameters = 18

There are 3 fixed effects ($f = 3$) and five observations in the "complete data" for each person ($T = 5$). Thus, there are a total of $f + T(T+1)/2 = 3 + 5(5+1)/2 = 18$ parameters. This is the end of the unrestricted model output.

Next follows the results for the homogeneous level-1 variance.

**Output for Random Effects Model with Homogeneous Level-1 Variance**

**Summary of the model specified**

**Level-1 Model**

$ATTIT_{mi}$ = $(IND1_{mi})$*$ATTIT_{1i}^*$ + $(IND2_{mi})$*$ATTIT_{2i}^*$ + $(IND3_{mi})$*$ATTIT_{3i}^*$ + $(IND4_{mi})$*$ATTIT_{4i}^*$ + $(IND5_{mi})$*$ATTIT_{5i}^*$

$ATTIT_{ti}^*$ = $\pi_{0i}$ + $\pi_{1i}$*$(AGE13_{ti})$ + $\pi_{2i}$*$(AGE13S_{ti})$ + $\varepsilon_{ti}$

**Level-2 Model**

$\pi_{0i} = \beta_{00} + r_{0i}$
$\pi_{1i} = \beta_{10} + r_{1i}$
$\pi_{2i} = \beta_{20} + r_{2i}$

$Var(\varepsilon_i) = Var(\boldsymbol{A}\boldsymbol{r}_i + e_i) = \boldsymbol{\Delta} = \boldsymbol{A\tau A'} + \sigma^2\boldsymbol{I}$

The above equation, written with subscripts and Greek letters, is

$$Var(Y^*) = \Delta = ATA' + \Sigma$$

where $\Sigma = \sigma^2 I_T$ .

**A**

| | | | |
|---|---|---|---|
| IND1 | 1.00000 | -2.00000 | 4.00000 |
| IND2 | 1.00000 | -1.00000 | 1.00000 |
| IND3 | 1.00000 | 0.00000 | 0.00000 |
| IND4 | 1.00000 | 1.00000 | 1.00000 |
| IND5 | 1.00000 | 2.00000 | 4.00000 |

The above matrix describes the design matrix on occasions one through five.

Iterations stopped due to small change in likelihood function

**Note:** The results below duplicate exactly the results produced by a standard HLM2 run using homogeneous level-1 variance.

**Final Results - Iteration 5**

| | Parameter | Standard Error |
|---|---|---|
| $\sigma^2$ | 0.02421 | 0.001672 |

$\tau$

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 0.04200 | 0.00808 | -0.00242 |
| AGE13,$r_1$ | 0.00808 | 0.00277 | -0.00012 |
| AGE13S,$r_2$ | -0.00242 | -0.00012 | 0.00049 |

**Standard errors of $\tau$**

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 0.00513 | 0.00127 | 0.00089 |
| AGE13,$r_1$ | 0.00127 | 0.00054 | 0.00024 |
| AGE13S,$r_2$ | 0.00089 | 0.00024 | 0.00025 |

**$\tau$ (as correlations)**

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 1.000 | 0.749 | -0.532 |
| AGE13,$r_1$ | 0.749 | 1.000 | -0.101 |
| AGE13S,$r_2$ | -0.532 | -0.101 | 1.000 |

**$\Delta$**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 0.03536 | 0.01388 | 0.01616 | 0.01801 | 0.01943 |
| IND2 | 0.01388 | 0.04870 | 0.03150 | 0.03488 | 0.03464 |
| IND3 | 0.01616 | 0.03150 | 0.06620 | 0.04766 | 0.04849 |
| IND4 | 0.01801 | 0.03488 | 0.04766 | 0.08056 | 0.06095 |
| IND5 | 0.01943 | 0.03464 | 0.04849 | 0.06095 | 0.09625 |

The $5 \times 5$ matrix above contains the five variance and ten covariance estimates implied by the "homogeneous level-1 variance" model.

**$\Delta$ (as correlations)**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 1.000 | 0.334 | 0.334 | 0.338 | 0.333 |
| IND2 | 0.334 | 1.000 | 0.555 | 0.557 | 0.506 |
| IND3 | 0.334 | 0.555 | 1.000 | 0.653 | 0.607 |
| IND4 | 0.338 | 0.557 | 0.653 | 1.000 | 0.692 |
| IND5 | 0.333 | 0.506 | 0.607 | 0.692 | 1.000 |

The value of the log-likelihood function at iteration 5 = 1.741132E+002

**Final estimation of fixed effects:**

| Fixed Effect | Coefficient | Standard error | *t*-ratio | Approx. *d.f.* | *p*-value |
|---|---|---|---|---|---|
| For INTRCPT1, $\pi_0$ | | | | | |
| INTRCPT2, $\beta_{00}$ | 0.327231 | 0.015306 | 21.379 | 238 | <0.001 |
| For AGE13 slope, $\pi_1$ | | | | | |
| INTRCPT2, $\beta_{10}$ | 0.064704 | 0.004926 | 13.135 | 238 | <0.001 |
| For AGE13S slope, $\pi_2$ | | | | | |
| INTRCPT2, $\beta_{20}$ | 0.000171 | 0.003218 | 0.053 | 238 | 0.958 |

**Statistics for the current model**

Deviance = -348.226421
Number of estimated parameters = 10

There are 3 fixed effects ($f$ = 3); the dimension of $\tau$ is 3, and a common $\sigma^2$ is estimated at level-1. Thus, there are a total of $f + r(r+1)/2 + 1 = 3 + 3(3+1)/2 + 1 = 10$ parameters.

This is the end of the output for the "homogeneous level-1 variance" model. Finally, the heterogeneous level-1 variance solution is listed.

**Output for Random Effects Model with Heterogeneous Level-1 Variance**

**Summary of the model specified**

**Level-1 Model**

$$\text{ATTIT}_{mi} = (\text{IND1}_{mi})*\text{ATTIT}_{1i}^{*} + (\text{IND2}_{mi})*\text{ATTIT}_{2i}^{*} + (\text{IND3}_{mi})*\text{ATTIT}_{3i}^{*} + (\text{IND4}_{mi})*\text{ATTIT}_{4i}^{*} + (\text{IND5}_{mi})*\text{ATTIT}_{5i}^{*}$$

$$\text{ATTIT}_{ti}^{*} = \pi_{0i} + \pi_{1i}*(\text{AGE13}_{ti}) + \pi_{2i}*(\text{AGE13S}_{ti}) + \varepsilon_{ti}$$

**Level-2 Model**

$$\pi_{0i} = \beta_{00} + r_{0i}$$
$$\pi_{1i} = \beta_{10} + r_{1i}$$
$$\pi_{2i} = \beta_{20} + r_{2i}$$

$Var(\varepsilon_i) = Var(\boldsymbol{A}\boldsymbol{r}_i + e_i) = \boldsymbol{\Delta} = \boldsymbol{A}*\boldsymbol{\tau}*\boldsymbol{A}' + diag(\sigma^2{}_1,...,\sigma^2{}_5)$

The above equation, written with subscripts and Greek letters, is

$$Var(Y^*) = ATA' + \Sigma$$

where $\Sigma = diag\left\{\sigma_t^2\right\}$, i.e. that is, $\Sigma$ is now a diagonal matrix with diagonal elements $\sigma_t^2$, the variance associated with occasion $t$, $t = 1, 2, \ldots, T$.

**A**

| | | | |
|---|---|---|---|
| IND1 | 1.00000 | -2.00000 | 4.00000 |
| IND2 | 1.00000 | -1.00000 | 1.00000 |
| IND3 | 1.00000 | 0.00000 | 0.00000 |
| IND4 | 1.00000 | 1.00000 | 1.00000 |
| IND5 | 1.00000 | 2.00000 | 4.00000 |

Iterations stopped due to small change in likelihood function

**Final Results - Iteration 8**

| | $\sigma^2$ | Standard Error |
|---|---|---|
| IND1 | 0.01373 | 0.005672 |
| IND2 | 0.02600 | 0.003296 |
| IND3 | 0.02685 | 0.003658 |
| IND4 | 0.02602 | 0.003633 |
| IND5 | 0.00275 | 0.007377 |

The five estimates above are the estimates of the level-1 variance for each time point.

$\tau$

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 0.04079 | 0.00736 | -0.00241 |
| AGE13,$r_1$ | 0.00736 | 0.00382 | 0.00025 |
| AGE13S,$r_2$ | -0.00241 | 0.00025 | 0.00106 |

**Standard errors of $\tau$**

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 0.00512 | 0.00124 | 0.00088 |
| AGE13,$r_1$ | 0.00124 | 0.00066 | 0.00042 |
| AGE13S,$r_2$ | 0.00088 | 0.00042 | 0.00030 |

**$\tau$ (as correlations)**

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 1.000 | 0.590 | -0.366 |
| AGE13,$r_1$ | 0.590 | 1.000 | 0.124 |
| AGE13S,$r_2$ | -0.366 | 0.124 | 1.000 |

$\mathbf{\Delta}$

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 0.03410 | 0.01707 | 0.01646 | 0.01851 | 0.02325 |
| IND2 | 0.01707 | 0.05165 | 0.03103 | 0.03322 | 0.03223 |
| IND3 | 0.01646 | 0.03103 | 0.06764 | 0.04574 | 0.04588 |
| IND4 | 0.01851 | 0.03322 | 0.04574 | 0.08208 | 0.06421 |
| IND5 | 0.02325 | 0.03223 | 0.04588 | 0.06421 | 0.08996 |

The $5 \times 5$ matrix above contains the estimates of five variances and ten covariances implied by the "heterogeneous level-1 variance" model.

**$\mathbf{\Delta}$ (as correlations)**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 1.000 | 0.407 | 0.343 | 0.350 | 0.420 |
| IND2 | 0.407 | 1.000 | 0.525 | 0.510 | 0.473 |
| IND3 | 0.343 | 0.525 | 1.000 | 0.614 | 0.588 |
| IND4 | 0.350 | 0.510 | 0.614 | 1.000 | 0.747 |
| IND5 | 0.420 | 0.473 | 0.588 | 0.747 | 1.000 |

The value of the log-likelihood function at iteration 8 = 1.816074E+002

**Final estimation of fixed effects:**

| Fixed Effect | Coefficient | Standard error | *t*-ratio | Approx. *d.f.* | *p*-value |
|---|---|---|---|---|---|
| For INTRCPT1, $\pi_0$ | | | | | |
| INTRCPT2, $\beta_{00}$ | 0.327646 | 0.015252 | 21.482 | 238 | <0.001 |
| For AGE13 slope, $\pi_1$ | | | | | |
| INTRCPT2, $\beta_{10}$ | 0.060864 | 0.004737 | 12.849 | 238 | <0.001 |
| For AGE13S slope, $\pi_2$ | | | | | |
| INTRCPT2, $\beta_{20}$ | -0.000541 | 0.003178 | -0.170 | 238 | 0.865 |

**Statistics for the current model**

Deviance = -363.214879
Number of estimated parameters = 14

There are 3 fixed effects ($f$ = 3), the dimension of $\tau$ is 3, and there are five observations intended for each person, each associated with a unique level-1 variance. Thus, there are a total of $f + r(r+1)/2 + T = 3 + 3(4)/2 + 5 = 14$ parameters.

**Summary of Model Fit**

| Model | Number of Parameters | Deviance |
|---|---|---|
| 1. Unrestricted | 18 | -378.26694 |
| 2. Homogeneous $\sigma^2$ | 10 | -348.22642 |
| 3. Heterogeneous $\sigma^2$ | 14 | -363.21488 |

| Model Comparison | $\chi^2$ | *d.f.* | *p*-value |
|---|---|---|---|
| Model 1 vs Model 2 | 30.04052 | 8 | $<0.001$ |
| Model 1 vs Model 3 | 15.05206 | 4 | 0.005 |
| Model 2 vs Model 3 | 14.98846 | 4 | 0.005 |

The model deviances are employed to evaluate the fits of the three models (unrestricted, homogeneous $\sigma^2$, and heterogeneous $\sigma^2$). Differences between deviances are distributed asymptotically as chi-square variates under the null hypothesis that the simpler model fits the data as well as the more complex model does. The results show that Model 1 fits better than does the homogeneous sigma_squared model $\chi^2$ = 30.04052, df = 8; it also fits better than does the heterogeneous sigma_squared model $\chi^2$ = 15.05206, df = 4.

In addition to the evaluation of models based on their fit to the data, the above results can be used to check the sensitivity of key inferences to alternative specifications of the variance-covariance structure. For instance, one could compare the mean and variance in the rate of change at age 13 obtained in Model 2 and Model 3 to assess how robust the results are to alternative plausible covariance specifications. The mean rate, $\gamma_{10}$, for Model 2 is 0.064704 (s.e. = 0.004926), and the variance, $\tau_{22}$, is 0.00277 (s.e. = 0.00054). The mean rate, G10, for Model 3 is 0.060864 (s.e. = 0.004737), and the variance, $\tau_{22}$, is 0.00382 (s.e. = 0.00066). The results are basically similar. See Raudenbush (2001) for a more detailed analysis of alternative covariance structures for polynomial models of individual growth and change using the same NYS data sets employed here for the illustrations.

Below are partial outputs for two random effect models.

**Output for Random Effects Model for Log-linear model for Level-1 Variance**

**Summary of the model specified**

**Level-1 Model**

$ATTIT_{mi} = (IND1_{mi})*ATTIT_{1i}^{*} + (IND2_{mi})*ATTIT_{2i}^{*} + (IND3_{mi})*ATTIT_{3i}^{*} + (IND4_{mi})*ATTIT_{4i}^{*} + (IND5_{mi})*ATTIT_{5i}^{*}$

$ATTIT_{ti}^{*} = \pi_{0i} + \pi_{1i}*(AGE13_{ti}) + \pi_{2i}*(AGE13S_{ti}) + \varepsilon_{ti}$

**Level-2 Model**

$\pi_{0i} = \beta_{00} + r_{0i}$
$\pi_{1i} = \beta_{10} + r_{1i}$
$\pi_{2i} = \beta_{20} + r_{2i}$

$Var(\varepsilon_i) = Var(\boldsymbol{A}\boldsymbol{r}_i + e_i) = \boldsymbol{\Delta} = \boldsymbol{A\tau A'} + diag(\sigma^2_1,\ldots,\sigma^2_5)$

The above equation, written with subscripts and Greek letters, is

$$Var(Y^*) = ATA' + \Sigma$$

where $\Sigma = diag(\sigma_t^2)$, and

$$\log(\sigma_t^2) = \alpha_0 + \alpha_1\,(EXPO)_t.$$

**A**

| | | | |
|---|---|---|---|
| IND1 | 1.00000 | -2.00000 | 4.00000 |
| IND2 | 1.00000 | -1.00000 | 1.00000 |
| IND3 | 1.00000 | 0.00000 | 0.00000 |
| IND4 | 1.00000 | 1.00000 | 1.00000 |
| IND5 | 1.00000 | 2.00000 | 4.00000 |

Iterations stopped due to small change in likelihood function

**Final results – Iteration 7**

| | Parameter | Standard Error |
|---|---|---|
| $\alpha_0$ | -3.72883 | 0.069238 |
| $\alpha_1$ | -1.43639 | 1.053241 |

| | $\sigma^2$ |
|---|---|
| IND1 | 0.02690 |
| IND2 | 0.02677 |
| IND3 | 0.02419 |
| IND4 | 0.02188 |
| IND5 | 0.02136 |

$\tau$

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 0.04255 | 0.00831 | -0.00257 |
| AGE13,$r_1$ | 0.00831 | 0.00277 | -0.00005 |
| AGE13S,$r_2$ | -0.00257 | -0.00005 | 0.00051 |

**Standard errors of $\tau$**

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 0.00517 | 0.00128 | 0.00089 |
| AGE13,$r_1$ | 0.00128 | 0.00054 | 0.00025 |
| AGE13S,$r_2$ | 0.00089 | 0.00025 | 0.00025 |

**$\tau$ (as correlations)**

| | | | |
|---|---|---|---|
| INTRCPT1,$r_0$ | 1.000 | 0.766 | -0.549 |
| AGE13,$r_1$ | 0.766 | 1.000 | -0.042 |
| AGE13S,$r_2$ | -0.549 | -0.042 | 1.000 |

**$\Delta$**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 0.03576 | 0.01267 | 0.01566 | 0.01782 | 0.01917 |
| IND2 | 0.01267 | 0.05095 | 0.03168 | 0.03516 | 0.03464 |
| IND3 | 0.01566 | 0.03168 | 0.06674 | 0.04829 | 0.04889 |
| IND4 | 0.01782 | 0.03516 | 0.04829 | 0.07909 | 0.06192 |
| IND5 | 0.01917 | 0.03464 | 0.04889 | 0.06192 | 0.09510 |

The $5 \times 5$ matrix above contains the variance and covariance estimates implied by the "log-linear" model for the level-1 variance.

**$\Delta$ (as correlations)**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 1.000 | 0.297 | 0.320 | 0.335 | 0.329 |
| IND2 | 0.297 | 1.000 | 0.543 | 0.554 | 0.498 |
| IND3 | 0.320 | 0.543 | 1.000 | 0.665 | 0.614 |
| IND4 | 0.335 | 0.554 | 0.665 | 1.000 | 0.714 |
| IND5 | 0.329 | 0.498 | 0.614 | 0.714 | 1.000 |

The value of the log-likelihood function at iteration 7 = 1.749582E+002

**Final estimation of fixed effects:**

| Fixed Effect | Coefficient | Standard error | *t*-ratio | Approx. *d.f.* | *p*-value |
|---|---|---|---|---|---|
| For INTRCPT1, $\pi_0$ | | | | | |
| INTRCPT2, $\beta_{00}$ | 0.328946 | 0.015379 | 21.390 | 238 | <0.001 |
| For AGE13 slope, $\pi_1$ | | | | | |
| INTRCPT2, $\beta_{10}$ | 0.064661 | 0.004923 | 13.135 | 238 | <0.001 |
| For AGE13S slope, $\pi_2$ | | | | | |
| INTRCPT2, $\beta_{20}$ | -0.000535 | 0.003222 | -0.166 | 238 | 0.869 |

**Statistics for the current model**

Deviance = -349.916489
Number of estimated parameters = 11

There are 3 fixed effects ($f = 3$), the dimension of τ is 3 ($r = 3$), and there is 1 intercept and 1 explanatory ($H = 1$) variable. Thus, there are a total of $f + r(r+1)/2 + 1 + H = 3 + 3(3+1)/2 + 1 + 1 = 11$ parameters.

Next are the results for the first-order auto-regressive model (Example: NYS4.MLM)

**Output for Random Effects Model First-order Autoregressive Model for Level-1 Variance**

**Summary of the model specified**

**Level-1 Model**

$$\mathrm{ATTIT}_{mi} = (\mathrm{IND1}_{mi})^*\mathrm{ATTIT}_{1i}^* + (\mathrm{IND2}_{mi})^*\mathrm{ATTIT}_{2i}^* + (\mathrm{IND3}_{mi})^*\mathrm{ATTIT}_{3i}^* + (\mathrm{IND4}_{mi})^*\mathrm{ATTIT}_{4i}^* + (\mathrm{IND5}_{mi})^*\mathrm{ATTIT}_{5i}^*$$

$$\mathrm{ATTIT}_{t\,i}^* = \pi_{0i} + \pi_{1i}^*(\mathrm{AGE13}_{t\,i}) + \pi_{2i}^*(\mathrm{AGE13S}_{t\,i}) + \varepsilon_{t\,i}$$

**Level-2 Model**

$$\pi_{0i} = \beta_{00} + r_{0i}$$
$$\pi_{1i} = \beta_{10}$$
$$\pi_{2i} = \beta_{20}$$

Note that $\beta_1$ and $\beta_2$ are specified as non-random due to the fact that the time-series is relatively short and therefore the data do not allow the estimation of both random slopes and an autocorrelation parameter.

$$Var(\varepsilon_i) = Var(\boldsymbol{A}\boldsymbol{r}_i + e_i) = \boldsymbol{\Delta} = \boldsymbol{A\tau A'} + \sigma^2\rho^{|t - t'|}$$

The above equation, written with subscripts and Greek letters, is

$$Var(Y^*) = ATA' + \Sigma$$

where

$$\Sigma = \sigma^2 \rho^{|t-t'|}.$$

**A**

| | |
|---|---|
| IND1 | 1.00000 |
| IND2 | 1.00000 |
| IND3 | 1.00000 |
| IND4 | 1.00000 |
| IND5 | 1.00000 |

Iterations stopped due to small change in likelihood function

**Final Results - Iteration 6**

| | Parameter | Standard Error |
|---|---|---|
| ρ | 0.39675 | 0.053849 |
| $\sigma^2$ | 0.04158 | 0.003582 |

Note that the maximum-likelihood estimate of $\hat{\rho}$ = 0.397 is much larger than its standard error (0.054), suggesting a significantly positive autocorrelation.

τ

| | |
|---|---|
| INTRCPT1,$r_0$ | 0.02427 |

**Standard error of τ**

| | |
|---|---|
| INTRCPT1,$r_0$ | 0.00450 |

**Δ**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 0.06585 | 0.04077 | 0.03081 | 0.02686 | 0.02530 |
| IND2 | 0.04077 | 0.06585 | 0.04077 | 0.03081 | 0.02686 |
| IND3 | 0.03081 | 0.04077 | 0.06585 | 0.04077 | 0.03081 |
| IND4 | 0.02686 | 0.03081 | 0.04077 | 0.06585 | 0.04077 |
| IND5 | 0.02530 | 0.02686 | 0.03081 | 0.04077 | 0.06585 |

The $5 \times 5$ matrix above contains the variance and covariance estimates implied by the "auto-correlation" model for the level-1 variance.

**Δ (as correlations)**

| | | | | | |
|---|---|---|---|---|---|
| IND1 | 1.000 | 0.619 | 0.468 | 0.408 | 0.384 |
| IND2 | 0.619 | 1.000 | 0.619 | 0.468 | 0.408 |
| IND3 | 0.468 | 0.619 | 1.000 | 0.619 | 0.468 |
| IND4 | 0.408 | 0.468 | 0.619 | 1.000 | 0.619 |
| IND5 | 0.384 | 0.408 | 0.468 | 0.619 | 1.000 |

The value of the log-likelihood function at iteration 6 = 1.471600E+002

**Final estimation of fixed effects:**

| Fixed Effect | Coefficient | Standard error | *t*-ratio | Approx. *d.f.* | *p*-value |
|---|---|---|---|---|---|
| For INTRCPT1, $\pi_0$ | | | | | |
| INTRCPT2, $\beta_{00}$ | 0.327579 | 0.015265 | 21.459 | 238 | <0.001 |
| For AGE13 slope, $\pi_1$ | | | | | |
| INTRCPT2, $\beta_{10}$ | 0.061428 | 0.004836 | 12.703 | 1076 | <0.001 |
| For AGE13S slope, $\pi_2$ | | | | | |
| INTRCPT2, $\beta_{20}$ | 0.000211 | 0.003373 | 0.062 | 1076 | 0.951 |

**Statistics for the current model**

Deviance = -294.319916
Number of estimated parameters = 6

**Summary of Model Fit**

| Model | Number of Parameters | Deviance |
|---|---|---|
| 1. Unrestricted | 18 | -378.26694 |
| 2. Homogeneous $\sigma^2$ | 5 | -229.01630 |
| 3. First order Autoregressive | 6 | -294.31992 |