

LISREL with mean structures

The LISREL model is now defined by the following three equations respectively:

$$\begin{aligned}\boldsymbol{\eta} &= \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}, \\ \mathbf{y} &= \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \\ x &= \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta},\end{aligned}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\tau}_y$, and $\boldsymbol{\tau}_x$ are vectors of constant intercept terms. As before, we assume that $\boldsymbol{\zeta}$ is uncorrelated with $\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}$ is uncorrelated with $\boldsymbol{\eta}$ and that $\boldsymbol{\delta}$ is uncorrelated with $\boldsymbol{\xi}$. We also assume, as before, that $E(\boldsymbol{\zeta}) = \mathbf{0}$, $E(\boldsymbol{\varepsilon}) = \mathbf{0}$, and $E(\boldsymbol{\delta}) = \mathbf{0}$, but it is not assumed that $E(\boldsymbol{\xi})$ and $E(\boldsymbol{\eta})$ are zero (E is the expected value operator). The mean of $\boldsymbol{\xi}$, $E(\boldsymbol{\xi})$, will be a parameter denoted by $\boldsymbol{\kappa}$. The mean of $\boldsymbol{\eta}$, $E(\boldsymbol{\eta})$, is obtained by taking the expectation of the first equation:

$$E(\boldsymbol{\eta}) = (\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa}).$$

By taking the expectations of the other two equations, we find the mean vectors of the observed variables to be

$$\begin{aligned}\boldsymbol{\mu}_y &= \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y(\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa}), \\ \boldsymbol{\mu}_x &= \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x\boldsymbol{\kappa}.\end{aligned}$$

In general, in a single population, all the mean parameters $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$ will not be identified without further conditions imposed. However, in simultaneous analysis of data from several groups, simple conditions (see Jöreskog & Sörbom (1985)) can be imposed to make all the mean parameters identified.

The LISREL model with mean structures introduces four new parameter matrices (actually vectors): $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$. These parameter matrices can be referred to in the same way as the other parameter matrices in LISREL. The LISREL notation and default forms for these are shown in the table below. Each of these parameter matrices is fixed at zero by default. They will be included in the model as soon as they are explicitly mentioned on the MO command. They can be declared either fixed (FI), free (FR), or invariant (IN); or with the same pattern (SP), same starting values (SS), or both (PS) as in the previous group.

The fit function for the extended LISREL model with mean parameters is defined as

$$F = \sum_{g=1}^G \frac{N_g}{N} F_g,$$

where

$$F_g = (\mathbf{s}^{(g)} - \boldsymbol{\sigma}^{(g)})' \mathbf{W}_{(g)}^{-1} (\mathbf{s}^{(g)} - \boldsymbol{\sigma}^{(g)}) + \left(\bar{\mathbf{z}}^{(g)} - \boldsymbol{\mu}^{(g)} \right)' \mathbf{V}_{(g)}^{-1} \left(\bar{\mathbf{z}}^{(g)} - \boldsymbol{\mu}^{(g)} \right)$$

and $\boldsymbol{\mu}^{(g)} = (\boldsymbol{\mu}_y^{(g)}, \boldsymbol{\mu}_x^{(g)})'$.

Table: Additional parameter matrices in LISREL

Name	Math symbol	Order	LISREL name	Possible modes*
TAU-Y	$\boldsymbol{\tau}_y$	NY×1	TY	FI, FR, IN, PS, SP, SS
TAU-X	$\boldsymbol{\tau}_x$	NX×1	TX	FI, FR, IN, PS, SP, SS
ALPHA	$\boldsymbol{\alpha}$	NE×1	AL	FI, FR, IN, PS, SP, SS
KAPPA	$\boldsymbol{\kappa}$	NK×1	KA	FI, FR, IN, PS, SP, SS

* Since these are vectors, Form is not relevant, only Mode is.

The second term in the equation above involves the sample mean vector $\bar{\mathbf{z}}^{(g)}$, the population mean vector $\boldsymbol{\mu}^{(g)}$, a function of parameters defined by the equations for $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$ given above, and the weight matrix $\mathbf{V}_{(g)}$ defined as:

$$\mathbf{V}_{(g)} = \mathbf{S}^{(g)} \quad \text{for ULS, GLS, WLS, DWLS}$$

$$\mathbf{V}_{(g)} = \hat{\Sigma}^{(g)} \quad \text{for ML}$$

It should be noted that if $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$ are all default, the second term in the expression for F_g is a constant, independent of parameters, in which case no mean structures are needed.

If the observed variables have a multivariate normal distribution the ML case defined above yields maximum likelihood estimates in the sense of maximizing the multinormal likelihood function. Under the same assumption, ML and GLS give asymptotically efficient estimators. The fit function for F may be justified under the more general assumption that $\bar{\mathbf{z}}^{(g)}$ and $\mathbf{S}^{(g)}$ are asymptotically uncorrelated. This holds, in particular, if the observed variables have no skewness.

Although the mean of a latent variable is undefined (not identified) in a single group, group differences in the means of latent variables can be estimated if the latent variables are on the same scale in all groups. Sörbom (1974) extended the classical model of factorial invariance, so that factor means could also be estimated. Sörbom's model is a LISREL Submodel 1 with mean structures defined as

$$\mathbf{x}^{(g)} = \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x \boldsymbol{\xi}^{(g)} + \boldsymbol{\delta}^{(g)}, \quad g = 1, 2, \dots, G,$$

with the mean of $\boldsymbol{\xi}^{(g)} = \boldsymbol{\kappa}^{(g)}$. To define the origins and the units of measurement of the $\boldsymbol{\xi}$ -factors, one can set $\boldsymbol{\kappa}^{(g)} = \mathbf{0}$ and fix one non-zero value in each column of $\boldsymbol{\Lambda}_x$. The parameters to be estimated are:

$\boldsymbol{\tau}_x, \boldsymbol{\Lambda}_x$, assumed to be invariant over groups

$\boldsymbol{\kappa}^{(1)}, \boldsymbol{\kappa}^{(2)}, \dots, \boldsymbol{\kappa}^{(g-1)}$, mean vectors of $\boldsymbol{\xi}$

$\Phi^{(1)}, \Phi^{(2)}, \dots, \Phi^{(G)}$, covariance matrices of ξ

$\Theta_{\delta}^{(1)}, \Theta_{\delta}^{(2)}, \dots, \Theta_{\delta}^{(G)}$, error covariance matrices

The error covariance matrices Θ_{δ} may be postulated to be invariant over groups, if desired. In most cases these matrices are diagonal.

In this example (Holzinger & Swineford (1939)) nine variables were selected to measure three latent factors: Space, Verbal and Memory. The groups consist of eighth-grade children from two schools in Chicago: the Pasteur and the Grant-White schools. The children from each school were divided into two groups according to whether they scored above or below the median on a speeded addition test. Thus the groups are:

1. Pasteur Low ($N_1 = 77$)
2. Pasteur High ($N_2 = 79$)
3. Grant-White Low ($N_3 = 74$)
4. Grant-White High ($N_4 = 71$)

The variables, correlations, standard deviations, and means are given in the two tables below. The standard deviations and means have been scaled so that the weighted estimate of the within groups covariance matrix is a correlation matrix.

Table: Nine psychological variables: correlations

Pasteur	Group 1 above diagonal; Group 2 below diagonal								
Test	1	2	3	4	5	6	7	8	9
Visual Perception	-	.32	.48	.28	.26	.40	.42	.12	.23
Cubes	.24	-	.33	.01	.01	.26	.32	.05	-.04
Paper Form Board	.23	.22	-	.06	.01	.10	.22	.03	.01
General Information	.43	.05	.23	-	.75	.60	.15	-.08	-.05
Sentence Completion	.35	.23	.18	.68	-	.63	.07	.06	.10
Word Classification	.36	.10	.11	.59	.66	-	.36	.19	.24
Figure Recognition	.22	.01	-.07	.09	.11	.12	-	.29	.19
Object-Number	-.02	-.01	-.13	.05	.08	.03	.19	-	.38
Number-Figure	.09	-.14	-.06	.16	.02	.12	.15	.29	-

Grant-White	Group 3 above diagonal; Group 4 below diagonal								
Test	1	2	3	4	5	6	7	8	9
Visual Perception	-	.34	.41	.38	.40	.42	.35	.16	.35
Cubes	.32	-	.21	.32	.16	.13	.27	.01	.27
Paper Form Board	.34	.18	-	.31	.24	.35	.30	.09	.09
General Information	.41	.24	.31	-	.69	.55	.17	.31	.34
Sentence Completion	.22	.16	.29	.62	-	.65	.20	.30	.27
Word Classification	.27	.20	.32	.57	.61	-	.31	.34	.27
Figure Recognition	.48	.31	.32	.18	.20	.29	-	.31	.38
Object-Number	.20	.01	.15	.06	.19	.15	.36	-	.38
Number-Figure	.42	-.14	.40	.11	.07	.18	.35	.44	-

Table: Nine psychological variables: means and standard deviations

Test	Standard deviations				Means			
	1	2	3	4	5	6	7	8
Visual Perception	1.06	0.96	0.95	1.03	4.20	4.30	4.29	4.21
Cubes	1.20	0.86	1.03	0.86	5.25	5.03	5.32	5.33
Paper Form Board	1.02	0.99	0.92	1.06	4.96	5.06	5.02	5.09
General Information	1.03	0.96	0.99	1.01	2.98	3.41	3.72	4.15
Sentence Completion	1.08	1.06	0.96	0.91	3.20	3.38	3.78	3.88
Word Classification	0.99	1.01	0.95	1.05	3.34	4.76	5.17	5.59
Figure Recognition	1.17	1.01	0.81	0.98	13.42	13.62	13.70	13.72
Object-Number	1.00	1.10	0.83	1.04	1.74	2.14	1.30	1.78
Number-Figure	1.04	1.00	0.88	1.07	2.10	2.16	1.87	2.44

The nine labels are stored in the file **EX101.LAB**. All the remaining data are stored in the file **EX101.DAT** in the order: correlations for group 1, standard deviations for group 1, means for group 1, correlations for group 2, etc. In both files we use FORTRAN formats to record the data. Each group of data begins with a format line. After the last right parenthesis in the format, one can write any text. This can be used to identify the particular part of the data for which the format is intended. After the format follows the data.

The model used here is that outline previously, with Λ_x and Θ_δ invariant over groups. The command file (**EX101.LIS** in the **LISREL Examples** folder) is shown below.

```
Holzinger-Swineford Data: LX and TD invariant: PASTEUR Low
DA NI=9 NG=4 NO=77
CM FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT;LA FI=EX101.LAB Rewind
MO NX=9 NK=3 TX=FR KA=FI
LK
Space Verbal Memory
PA LX
3(1 0 0) 3(0 1 0) 3(0 0 1)
FI LX 1 1 LX 4 2 LX 7 3
VA 1 LX 1 1 LX 4 2 LX 7 3
OU TV SS ND=2
Holzinger-Swineford Data: LX and TD invariant: PASTEUR High
DA NO=79
CM FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT;LA FI=EX101.LAB Rewind
MO LX=IN TX=IN KA=FR TD=IN
LK
Space Verbal Memory
OU
Holzinger-Swineford Data: LX and TD invariant: GRANT-WHITE Low
DA NO=74
CM FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT;LA FI=EX101.LAB Rewind
MO
LK
Space Verbal Memory
OU
Holzinger-Swineford Data: LX and TD invariant: GRANT-WHITE High
DA NO=71
CM FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT;LA FI=EX101.LAB Rewind
MO
LK
Space Verbal Memory
OU
```

The first line for each group is a title line. The DA command for the first group specifies the number of groups: NG = 4. The next line actually contains three commands, separated by semicolons. These commands tell the program to read the correlation (KM), the standard deviations (SD), and the means (ME), in that order. As the correlation matrix in the data file has been recorded with one row per line, the option SY on the KM command is essential; otherwise, the program would expect the correlations as one long line. The file **EX101.LAB** will be rewound after it has been read, so that the same labels may be read again for the next group. Note that the rewind (RE) option only applies to the file **EX101.LAB**, not to the file **EX101.DAT**.

The next five command specify the model for the first group. The model includes the parameter matrices τ_x , Λ_x , Φ , and Θ_δ . LX, PH, and TD are default, TX is declared free for the first group and is declared invariant for the other groups, and KA is specified as fixed for the first group and free for the others. The OU command for the first group requests *t*-values (TV), standardized solution (SS), and 2 decimals (ND = 2). Note that this information is automatically carried onto the OU command for groups 2 – 4. Similarly, the information on the MO command for the second group is automatically carried onto the MO commands for groups 3 and 4. For the same reason, only the NO keyword needs to be specified on the DA command for groups 2, 3, and 4. The general rule is that options and keywords that are the same as in the previous group need not be specified.

Table: Maximum likelihood estimates for nine psychological variables with factor means (solution standardized to common metric)

Factor loadings:

Test	Factors			Error variance	Intercept
	Space	Verbal	Memory		
Visual Perception	0.72	0	0	0.28	4.20
Cubes	0.42	0	0	0.82	5.20
Paper Form Board	0.52	0	0	0.73	5.00
General Information	0	0.83	0	0.34	3.06
Sentence Completion	0	0.78	0	0.32	3.08
Word Classification	0	0.78	0	0.43	4.51
Figure Recognition	0	0	0.46	0.75	13.56
Object-Number	0	0	0.55	0.70	1.68
Number-Figure	0	0	0.61	0.63	2.07

Factor covariance matrices:

School	Factor	Low level			High level		
	Space	1.37			0.72		
Pasteur	Space	0.44	1.12	0.51		1.03	
	Memory	0.58	0.24	1.16	0.04	0.20	0.92
	Space	0.90			1.02		
Grant-White	Space	0.64	0.92	0.54		0.92	
	Memory	0.56	0.53	0.58	0.96	0.33	1.33

Factor means:

Group	Space	Verbal	Memory
Pasteur Low	0.00	0.00	0.00
Pasteur High	0.05	0.32	0.18
Grant-White Low	0.09	0.70	-0.14
Grant-White High	0.06	1.02	0.20