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The LISREL model

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1. The general LISREL model

The LISREL model, in its most general form, consists of two parts: the measurement model and the structural equation model.

The **measurement model** specifies how latent variables or hypothetical constructs depend upon or are indicated by the observed variables. It describes the measurement properties (reliabilities and validities) of the observed variables.

The **structural equation model** specifies the causal relationships among the latent variables, describes the causal effects, and assigns the explained and unexplained variance.

Recall that the full LISREL model for single samples is defined, for deviation about the mean, by the following three equations:

The structural equation model:

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\varsigma} \qquad (1)$$

The measurement model for **y**:

$$\mathbf{y} = \boldsymbol{\Lambda}_y \boldsymbol{\eta} + \boldsymbol{\varepsilon} \qquad (2)$$

The measurement model for **x**:

$$\mathbf{x} = \boldsymbol{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta} \qquad (3)$$

The terms in these models are defined as follows:

y is a $p \times 1$ vector of observed response or outcome variables.

x is a $q \times 1$ vector of predictors, covariates, or input variables.

$\boldsymbol{\eta}$ is an $m \times 1$ random vector of latent dependent, or endogenous, variables

$\boldsymbol{\xi}$ is an $n \times 1$ random vector of latent independent, or exogenous, variables

$\boldsymbol{\varepsilon}$ is a $p \times 1$ vector of measurement errors in **y.**

$\boldsymbol{\delta}$ is a $q \times 1$ vector of measurement errors in **x**.

$\boldsymbol{\Lambda}_y$ is a $p \times m$ matrix of coefficients of the regression of **y** on $\boldsymbol{\eta}$.

$\boldsymbol{\Lambda}_x$ is a $q \times n$ matrix of coefficients of the regression of **x** on $\boldsymbol{\xi}$.

$\boldsymbol{\Gamma}$ is an $m \times n$ matrix of coefficients of the ξ-variables in the structural relationship.

B is an $m \times m$ matrix of coefficients of the η-variables in the structural relationship. **B** has zeros in the diagonal, and **I** – **B** is required to be non-singular.

$\boldsymbol{\varsigma}$ is a $m \times 1$ vector of equation errors (random disturbances) in the structural relationship between $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$.

Assumptions

The random components in the LISREL model are assumed to satisfy the following minimal assumptions:

$\boldsymbol{\varepsilon}$ is uncorrelated with $\boldsymbol{\eta}$

$\boldsymbol{\delta}$ is uncorrelated with $\boldsymbol{\xi}$

$\boldsymbol{\varsigma}$ is uncorrelated with $\boldsymbol{\xi}$

$\boldsymbol{\varsigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ are mutually uncorrelated.

Covariance matrices:

$Cov(\boldsymbol{\xi}) = \boldsymbol{\Phi}(n \times n)$ $\quad$ $Cov(\boldsymbol{\varsigma}) = \boldsymbol{\Psi}(m \times m)$

$Cov(\boldsymbol{\varepsilon}) = \boldsymbol{\Theta}_\varepsilon (p \times p)$ $\quad$ $Cov(\boldsymbol{\delta}) = \boldsymbol{\Theta}_\delta (q \times q)$

The covariance matrix of the observations as implied by the LISREL model

The assumptions in the previous section imply the following form for the covariance matrix of the observed variables:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Lambda}_y \mathbf{A}\left(\boldsymbol{\Gamma\Phi\Gamma}' + \boldsymbol{\Psi}\right)\mathbf{A}'\boldsymbol{\Lambda}_y' + \boldsymbol{\Theta}_\varepsilon & \boldsymbol{\Lambda}_y \mathbf{A}\boldsymbol{\Gamma\Phi\Lambda}_x' \\ \boldsymbol{\Lambda}_x \boldsymbol{\Phi\Gamma}'\mathbf{A}'\boldsymbol{\Lambda}_y' & \boldsymbol{\Lambda}_x \boldsymbol{\Phi\Lambda}_x' + \boldsymbol{\Theta}_\delta \end{bmatrix} \tag{4}$$

where $\mathbf{A} = (\mathbf{I} - \mathbf{B})^{-1}$.

2. The extended LISREL model

The assumptions of the general LISREL model will now be relaxed and the model will be extended to include four new parameter matrices in addition to the previous eight. These new parameter matrices contain intercept terms in the relationships and mean values of the latent variables.

The LISREL model is now defined by the following three equations corresponding to (1), (2), and (3) respectively.

$$\boldsymbol{\eta} = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\varsigma}, \tag{4}$$

$$\mathbf{y} = \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \tag{5}$$

$$\mathbf{x} = \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta}, \tag{6}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\tau}_y$, and $\boldsymbol{\tau}_x$ are vectors of constant intercept terms. As before, we assume that $\boldsymbol{\varsigma}$ is uncorrelated with $\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}$ is uncorrelated with $\boldsymbol{\eta}$, and that $\boldsymbol{\delta}$ is uncorrelated with $\boldsymbol{\xi}$. We also assume, as before, that $E\left(\boldsymbol{\varsigma}\right) = \mathbf{0}$, $E\left(\boldsymbol{\varepsilon}\right) = \mathbf{0}$, and $E\left(\boldsymbol{\delta}\right) = \mathbf{0}$, but it is not assumed that $E\left(\boldsymbol{\xi}\right)$ and $E\left(\boldsymbol{\eta}\right)$ are zero (E is the expected value operator). The mean of $\boldsymbol{\xi}$, $E\left(\boldsymbol{\xi}\right)$, will be a parameter denoted by $\boldsymbol{\kappa}$. The mean of $\boldsymbol{\eta}$, $E\left(\boldsymbol{\eta}\right)$, is obtained by taking the expectation of (4):

$$E\left(\boldsymbol{\eta}\right) = (\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa}). \tag{7}$$

By taking the expectations of (5) and (6), we find the mean vectors of the observed variables to be

$$\boldsymbol{\mu}_y = \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y(\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa}), \tag{8}$$

$$\boldsymbol{\mu}_x = \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x\boldsymbol{\kappa}. \tag{9}$$

In general, in a single population, all the mean parameters $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$ will not be identified without further conditions imposed. However, in simultaneous analysis of data from several groups, simple conditions (see Jöreskog & Sörbom, 1985) can be imposed to make all the mean parameters identified.

The LISREL model with mean structures introduces four new parameter matrices (actually vectors): $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$.

These parameter matrices can be referred to in the same way as the other parameter matrices in LISREL. The LISREL notation and default forms for these are shown in Table 1. Each of these parameter matrices is fixed at zero by default. *They will be included in the model as soon as they are explicitly mentioned on the MO command.* They can be declared either fixed (FI), free (FR), or invariant (IN); or with the same pattern (SP), same starting values (SS), or both (PS) as in the previous group.

The fit function for the extended LISREL model with mean parameters is defined as

$$F = \sum_{g=1}^{G} \frac{N_g}{N} F_g, \tag{10}$$

where

$$F_g = (\mathbf{s}^{(g)} - \boldsymbol{\sigma}^{(g)})' \mathbf{W}_{(g)}^{-1} (\mathbf{s}^{(g)} - \boldsymbol{\sigma}^{(g)}) + \left(\bar{\mathbf{z}}^{(g)} - \boldsymbol{\mu}^{(g)} \right)' \mathbf{V}_{(g)}^{-1} \left(\bar{\mathbf{z}}^{(g)} - \boldsymbol{\mu}^{(g)} \right) \tag{11}$$

and $\boldsymbol{\mu}^{(g)} = (\boldsymbol{\mu}_y^{(g)}, \boldsymbol{\mu}_y^{(g)})'$.

Table 1: Additional parameter matrices in LISREL

Name	Math symbol	Order	LISREL name	Possible modes*
TAU-Y	$\boldsymbol{\tau}_y$	NY× 1	TY	FI, FR, IN, PS, SP, SS
TAU-X	$\boldsymbol{\tau}_x$	NX× 1	TX	FI, FR, IN, PS, SP, SS
ALPHA	$\boldsymbol{\alpha}$	NE× 1	AL	FI, FR, IN, PS, SP, SS
KAPPA	$\boldsymbol{\kappa}$	NK× 1	KA	FI, FR, IN, PS, SP, SS

The first term in (11) is the same as

$$F(\boldsymbol{\theta}) = (\mathbf{s} - \boldsymbol{\sigma})' \mathbf{W}^{-1} (\mathbf{s} - \boldsymbol{\sigma})$$
$$= \sum_{g=1}^{k} \sum_{h=1}^{g} \sum_{i=1}^{k} \sum_{j=1}^{i} w^{gh,ij} (s_{gh} - \sigma_{gh})(s_{ij} - \sigma_{ij}).$$

The second term involves the same mean vector $\bar{\mathbf{z}}^{(g)}$, the population mean vector $\boldsymbol{\mu}^{(g)}$, a function of parameters by (8) and (9), and the weight matrix $\mathbf{V}_{(g)}$ defined as

$$\mathbf{V}_{(g)} = \mathbf{S}^{(g)} \text{ for ULS, GLS, WLS, DWLS}$$
$$\mathbf{V}_{(g)} = \hat{\boldsymbol{\Sigma}}^{(g)} \text{ for ML}$$

It should be noted that if $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$ are all default, the second term in (11) is a constant, independent of parameters, in which case the problem reduces to one where not mean structures are needed.

If the observed variables have a multivariate normal distribution the ML case defined above yields maximum likelihood estimates in the sense of maximizing the multinormal likelihood function. Under the same assumption, ML and GLS give asymptotically efficient estimators. The fit function (10) may be justified under the more general assumption that $\bar{\mathbf{z}}^{(g)}$ and $\mathbf{S}^{(g)}$ are asymptotically uncorrelated. This holds, in particular, if the observed variables have no skewness.

3. References

Jöreskog, K.G. & Sörbom, D. (1985). Simultaneous analysis of longitudinal data from several cohorts. Pp. 323-341 in W.M. Mason and S.E. Fienberg (Eds.): *Cohort analysis in social research: Beyond the identification problem*. New York:

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