Econometric models

Unlike the previous example, where the system of equations was recursive, *econometric models* are usually *non-recursive* or so-called *interdependent systems*. Data for econometric models are often in the form of *time series* and the models are usually *dynamic* in the sense that elements of time play important roles in the model. Another characteristic of econometric models is that they often contain definitional equations or *identities*. These are exact relationships without disturbance terms and with no unknown parameters to be estimated.

Although the identities are not subject to estimation, they must nevertheless be specified in the command file in order to define which variables are *independent* (*exogenous*) and which are *jointly dependent* (*endogenous*). If the identities are omitted, the program will estimate the behavioral equations by ordinary least squares (OLS), yielding inconsistent (biased) estimates.

The following example illustrates the formulation of a non-recursive econometric model, including the identity relations. LISREL is then used to fit the model, subject to the identities, to a sample of time series data.

Klein's Model I (Klein (1950); Goldberger (1964); Theil (1971)) illustrates a classical econometric model that has been used extensively as a benchmark problem for studying econometric methods. It is an eight-equation system based on annual business data for the United States in the period between the two world wars.

The endogenous variables are:

C_t = Aggregate Consumption (y_1)

I_t = Net Investment (y_2)

W_t^* = Private Wage Bill (y_3)

P_t = Total Profits (y_4)

Y_t = Total Income (y_5)

K_t = End-of-Year Capital Stock (y_6)

W_t = Total Wage Bill (y_7)

E_t = Total Production of Private Industry (y_8)

The predetermined variables are the exogenous variables:

W_t^{**} = Government Wage Bill (x_1)
T_t = Taxes (x_2)
G_t = Government Non-Wage Expenditures (x_3)
A_t = Time in Years from 1931 (x_4)

In addition, the lagged endogenous variables are:

$$P_{t-1}(x_5),\ K_{t-1}(x_6),\ \text{and}\ E_{t-1}(x_7).$$

All variables except A_t are in billions of 1934 dollars.

The three behavioral equations of Klein's Model I are:

$$\begin{aligned} C_t &= a_1 P_t + a_2 P_{t-1} + a_3 W_t + \xi_1 \\ I_t &= b_1 P_t + b_2 P_{t-1} + b_3 K_{t-1} + \xi_2 \\ W_t^* &= c_1 E_t + c_2 E_{t-1} + c_3 A_t + \xi_3. \end{aligned}$$

In addition to these stochastic equations the model includes five identities:

$$\begin{aligned} P_t &= Y_t - W_t \\ Y_t &= C_t + I_t + G_t - T_t \\ K_t &= K_{t-1} + I_t \\ W_t &= W_t^* + W_t^{**} \\ E_t &= Y_t + T_t - W_t^{**} \end{aligned}$$

The model can be formulated as a LISREL model with p = 8 and q = 7 and with $\mathbf{B}$, $\mathbf{\Gamma}$, and $\mathbf{\Psi}$ as:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & a_1 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$
\mathbf{\Gamma} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b_2 & b_3 & 0 \\
0 & 0 & 0 & 0 & c_3 & 0 & 0 & c_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{\Psi} = \begin{bmatrix}
\psi_{11} & & & & & & & \\
\psi_{21} & \psi_{22} & & & & & & \\
\psi_{31} & \psi_{32} & \psi_{33} & & & & & \\
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

As a consequence of the identities in the model, the last five rows of $\mathbf{B}$, $\mathbf{\Gamma}$, and $\mathbf{\Psi}$ do not contain any parameters to be estimated. Another consequence of the identities is that both $\mathbf{\Sigma}$ and $\mathbf{S}$ will be singular if all the 15 variables are included in the model. As a result, it is impossible to use the ML method or GLS method since they require either $\mathbf{\Sigma}$ or $\mathbf{S}$ to be positive definite. However, by eliminating the five redundant *y*-variables, ML estimation is possible as shown below, after a preliminary IV and ULS analysis. Ridge estimation is also illustrated.

Unweighted least squares:

The following command file (**EX46A.LIS**) performs the ULS fitting and gives the IV estimates as a by-product. The labels and the raw data with their formats are stored in the file **EX46.DAT**:

```
KLEIN'S MODEL I ESTIMATED BY IV AND ULS
DA NI=15 NO=21
LA FI=EX46.DAT
RA FI=EX46.DAT
SE
1 4 3 10 14 11 13 12 7 8 15 9 2 5 6
MO NY=8 NX=7 BE=FU GA=FI PS=SY,FI
FR BE(1,4) BE(1,7) BE(2,4) BE(3,8)
FR GA(1,5) GA(2,5) GA(2,6) GA(3,4) GA(3,7)
FR PS(1,1)-PS(3,3)
VA 1 BE(4,5) BE(5,1) BE(5,2) BE(6,2) BE(7,3) BE(8,5) GA(5,3) GA(6,6) GA(7,1)
GA(8,2)
```

```
VA -1 BE(4,7) GA(5,2) GA(8,1)
OU ME=UL AD=OFF
```

The variables in the data files are not in the same order as in the model. The selection command puts them in the order $y_1, y_2, ..., y_8, x_1, x_2, ..., x_7$. The only parameter matrices needed are $\mathbf{B}$, $\mathbf{\Gamma}$, and $\mathbf{\Psi}$. $\mathbf{B}$ is declared full on the MO command and is fixed by default. $\mathbf{\Gamma}$ is declared fixed on the MO command and is full by default. $\mathbf{\Psi}$ is declared fixed on the MO command and is symmetric by default. After the MO command the free elements and the values of the non-zero fixed elements in $\mathbf{B}$ and $\mathbf{\Gamma}$ are defined. The FR and VA commands contain these definitions. The admissibility check is set to OFF in the OU command because $\mathbf{\Psi}$ has fixed diagonal zeros.

The results of the ULS run gives two different sets of estimates for the structural parameters: IV and ULS estimates. The estimates are shown in the table above.

Ridge estimates:

To obtain *ridge estimates* of the model parameters, merely delete ME = UL from the OU command (**EX46B.LIS**). The program will then attempt to perform the default maximum likelihood estimation. However, the covariance matrix is singular, so ML estimation is not possible. The program will therefore automatically invoke the ridge option with ridge constant 0.001. This gives the estimates shown in the table above (MLR).

Maximum likelihood estimation:

As already noted, the covariance matrices $\mathbf{\Sigma}$ and $\mathbf{S}$ ass well as the data matrix itself are singular because of the five identities in the model. The rank of these matrices is not 15 but 10. It is possible to solve the identities for the redundant variables P_t, Y_t, K_t, W_t and E_t in terms of the other 10 variables and substitute these solutions into the behavioral equations. This results in a system with three *y*-variables and seven *x*-variables and with coefficients which are linear combinations of the structural parameters.

Although it is possible to estimate the model in this form, this rather complicated approach is not necessary. LISREL will estimate the model directly if all the *y*-variables are treated as η-variables, of which only the first 3 are observed. The LISREL specification for this purpose is:

$$\mathbf{y}' = \begin{bmatrix} C_t & I_t & W_t^* \end{bmatrix}$$
$$\boldsymbol{\eta}' = \begin{bmatrix} C_t & I_t & W_t^* & P_t & Y_t & K_t & W_t & E_t \end{bmatrix}$$
$$\mathbf{x}'\mathbf{\Xi}' = \begin{bmatrix} W & T_t & G_t & A_t & P_{t-1} & K_{k-1} & E_{t-1} \end{bmatrix}.$$

The $\mathbf{\Lambda}_y$-matrix is:

$$\mathbf{\Lambda}_y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This the matrix form called **IZ**, while $\mathbf{B}$, $\mathbf{\Gamma}$, and $\mathbf{\Psi}$ are the same as before. The command file for the ML solution is (**EX46C.LIS**):

```
KLEIN'S MODEL I ESTIMATED BY TSLS AND ML
DA NI=15 NO=21
LA FI=EX46.DAT
RA FI=EX46.DAT
SE
1 4 3 7 8 15 9 2 5 6 /
MO NY=3 NE=8 NX=7 FI LY=IZ BE=FU GA=FI PS=SY,FI TE=ZE
FR BE(1,4) BE(1,7) BE(2,4) BE(3,8)
FR GA(1,5) GA(2,5) GA(2,6) GA(3,4) GA(3,7)
FR PS(1,1)-PS(3,3)
VA  1  BE(4,5)  BE(5,1)  BE(5,2)  BE(6,2)  BE(7,3)  BE(8,5)  GA(5,3)  GA(6,6)  GA(7,1)
GA(8,2)
VA -1 BE(4,7) GA(5,2) GA(8,1)
ST 5 PS(1,1) PS(2,2) PS(3,3)
OU NS AD=OFF IT=80
```

The input data are the same as before. But in this case NY = 3 and NX = 7, so only 10 variables are selected (note that the data on P_t, Y_t, K_t, W_t and E_t are not used). The MO command specifies NE = 8.

Since only NK is default on the MO command one must use FI to specify Fixed-x and LY = IZ and TE = ZE to specify that $y' = [\eta_1 \quad \eta_2 \quad \eta_3]$. Otherwise, the MO, FR and VA commands are the same as in the previous run.

One further complication arises. Since there are more η-variables than *y*-variables the assumption for the starting value algorithm is not fulfilled. The user must therefore provide such starting values that $\mathbf{\Sigma}$ becomes positive definite initially. This is accomplished simply by putting positive values in the first three diagonal elements of $\mathbf{\Psi}$, as shown by the ST command in the command file above. It is also recommended that the option NS be inserted on the OU command. This tells the program to use the steepest descent method to improve the starting point before the actual minimization of the fit function begins.

```
LISREL Estimates (Maximum Likelihood)

         LAMBDA-Y

               ETA 1      ETA 2      ETA 3      ETA 4      ETA 5      ETA 6
            --------   --------   --------   --------   --------   --------
       C       1.000        - -        - -        - -        - -        - -
       I         - -      1.000        - -        - -        - -        - -
      W*         - -        - -      1.000        - -        - -        - -
```

LAMBDA-Y

	ETA 7	ETA 8
C	- -	- -
I	- -	- -
W*	- -	- -

BETA

	ETA 1	ETA 2	ETA 3	ETA 4	ETA 5	ETA 6
ETA 1	- -	- -	- -	-0.232	- -	- -
				(0.312)		
				-0.745		
ETA 2	- -	- -	- -	-0.801	- -	- -
				(0.491)		
				-1.630		
ETA 3	- -	- -	- -	- -	- -	- -
ETA 4	- -	- -	- -	- -	1.000	- -
ETA 5	1.000	1.000	- -	- -	- -	- -
ETA 6	- -	1.000	- -	- -	- -	- -
ETA 7	- -	- -	1.000	- -	- -	- -
ETA 8	- -	- -	- -	- -	1.000	- -

BETA

	ETA 7	ETA 8
ETA 1	0.802	- -
	(0.036)	
	22.340	
ETA 2	- -	- -
ETA 3	- -	0.234
		(0.049)
		4.796
ETA 4	-1.000	- -
ETA 5	- -	- -
ETA 6	- -	- -
ETA 7	- -	- -
ETA 8	- -	- -

GAMMA

	W**	T	G	A	P-1	K-1
ETA 1	- -	- -	- -	- -	0.386	- -
					(0.217)	
					1.774	
ETA 2	- -	- -	- -	- -	1.052	-0.148
					(0.352)	(0.030)
					2.984	-4.961
ETA 3	- -	- -	- -	0.235	- -	- -
				(0.035)		
				6.807		
ETA 4	- -	- -	- -	- -	- -	- -
ETA 5	- -	-1.000	1.000	- -	- -	- -
ETA 6	- -	- -	- -	- -	- -	1.000
ETA 7	1.000	- -	- -	- -	- -	- -
ETA 8	-1.000	1.000	- -	- -	- -	- -

GAMMA

	E-1
ETA 1	- -
ETA 2	- -
ETA 3	0.285
	(0.045)
	6.297
ETA 4	- -
ETA 5	- -
ETA 6	- -
ETA 7	- -
ETA 8	- -

Covariance Matrix of ETA and KSI

	ETA 1	ETA 2	ETA 3	ETA 4	ETA 5	ETA 6
ETA 1	41.415					
ETA 2	7.753	9.036				
ETA 3	37.025	8.009	34.365			
ETA 4	14.850	8.970	13.631	12.771		
ETA 5	60.254	15.381	54.919	26.263	91.794	
ETA 6	36.534	2.427	33.908	7.124	45.930	94.194
ETA 7	45.404	6.411	41.288	13.492	65.532	38.806
ETA 8	59.768	16.496	54.776	26.484	90.932	46.322
W**	8.379	-1.599	6.923	-0.139	10.613	4.898
T	7.893	-0.484	6.780	0.083	9.750	5.290
G	18.980	-1.892	16.664	2.525	25.910	12.260
A	28.321	-5.000	23.706	0.000	35.606	24.530

P-1	14.733	7.719	14.332	10.531	24.153	13.881
K-1	28.781	-6.610	25.898	-1.846	30.549	91.768
E-1	48.726	9.930	45.355	19.278	71.181	52.016

Covariance Matrix of ETA and KSI

	ETA 7	ETA 8	W**	T	G	A
ETA 7	52.040					
ETA 8	64.448	92.250				
W**	10.752	9.672	3.829			
T	9.668	10.990	2.888	4.127		
G	23.385	25.657	6.720	6.468	15.289	
A	35.606	32.506	11.900	8.800	21.085	38.500
P-1	13.621	25.280	-0.711	0.416	2.117	-0.595
K-1	32.395	29.826	6.497	5.774	14.152	29.530
E-1	51.903	72.151	6.548	7.519	20.044	24.780

Covariance Matrix of ETA and KSI

	P-1	K-1	E-1
P-1	16.226		
K-1	6.161	98.377	
E-1	30.047	42.086	79.541

PHI

	W**	T	G	A	P-1	K-1
W**	3.829					
T	2.888	4.127				
G	6.720	6.468	15.289			
A	11.900	8.800	21.085	38.500		
P-1	-0.711	0.416	2.117	-0.595	16.226	
K-1	6.497	5.774	14.152	29.530	6.161	98.377
E-1	6.548	7.519	20.044	24.780	30.047	42.086

PHI

	E-1
E-1	79.541

PSI

	ETA 1	ETA 2	ETA 3	ETA 4	ETA 5	ETA 6
ETA 1	2.209					
	(1.950)					

```
              1.133

  ETA 2       4.073     13.410
             (4.083)    (8.754)
              0.998      1.532

  ETA 3       0.506      4.050      1.891
             (0.796)    (1.884)    (0.761)
              0.636      2.150      2.484

  ETA 4        - -        - -        - -        - -
  ETA 5        - -        - -        - -        - -        - -
  ETA 6        - -        - -        - -        - -        - -        - -
  ETA 7        - -        - -        - -        - -        - -        - -
  ETA 8        - -        - -        - -        - -        - -        - -


       PSI

              ETA 7      ETA 8
            --------   --------
  ETA 7        - -
  ETA 8        - -        - -


       Squared Multiple Correlations for Structural Equations

              ETA 1      ETA 2      ETA 3      ETA 4      ETA 5      ETA 6
            --------   --------   --------   --------   --------   --------
              0.919      0.592      0.862      1.000      1.000      1.000

       Squared Multiple Correlations for Structural Equations

              ETA 7      ETA 8
            --------   --------
              1.000      1.000

 NOTE: R² for Structural Equations are Hayduk's (2006) Blocked-Error R²
```

This run gives the TSLS and ML estimates of the structural parameters shown in columns 3 and 6 of the table of results below. It is apparent that ML and ridge estimates are considerably different from the IV, TSLS, and ULS estimates, which are closer to each other.

The ML estimates for econometric models of the form considered in this section are sometimes called FIML (Full Information Maximum Likelihood) estimates or FILGRV (Full Information Least Generalized Residual Variance) estimates. It has been shown that these estimates minimize the generalized variance of the reduced form residuals, i.e., the determinant of the reduced form residual covariance matrix (see Jöreskog, 1973). The ML estimates can therefore be justified without the assumption of normality. The user should be warned not to rely heavily on the standard errors and/or the χ^2-measure of fit, since they depend on both the assumption of normality and a large sample. Also, because autocorrelation is often present in time series data, the assumption of independent observations is questionable.

Table: Parameter estimates for Klein's Model I

Parameter	IV	TSLS	ULS	MLR	ML
a_1	0.04	0.02	0.04	-0.04	-0.23
a_2	0.19	0.22	0.21	0.29	0.39
a_3	0.82	0.81	0.81	0.79	0.80
b_1	0.04	0.15	0.05	-0.45	-0.80
b_2	0.69	0.62	0.68	0.98	1.05
b_3	-0.17	-0.16	-0.17	-0.18	-0.15
c_1	0.39	0.44	0.39	0.31	0.23
c_2	0.20	0.15	0.20	0.25	0.29
c_3	0.14	0.13	0.15	0.19	0.24