Scientific Software International

Confidence interval estimates

The extended LISREL model

In the extended LISREL model (Jöreskog and Sörbom 1999), the three sets of relationships between the observed and latent variables are given by

$$\mathbf{y} = \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y \boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

$$\mathbf{x} = \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}$$

$$\boldsymbol{\eta} = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}$$

where the elements of $\mathbf{y}$ denote p_y observed indicators of the m_η endogenous latent variables $\boldsymbol{\eta}$, the elements of $\mathbf{x}$ denote p_x observed indicators of the m_ξ exogenous latent variables $\boldsymbol{\xi}$, the elements of $\boldsymbol{\varepsilon}$ denote p_y measurement errors, the elements of $\boldsymbol{\delta}$ denote p_x measurement errors, the elements of $\boldsymbol{\zeta}$ denote m_η error variables, the elements of $\boldsymbol{\tau}_y$ are p_y intercepts, the elements of $\boldsymbol{\tau}_x$ are p_x intercepts, the elements of $\boldsymbol{\alpha}$ are m_η intercepts, the elements of $\boldsymbol{\Lambda}_y$ are $p_y \times m_\eta$ measurement weights, the elements of $\boldsymbol{\Lambda}_x$ are $p_x \times m_\xi$ measurement weights, the elements of $\mathbf{B}$ are $m_\eta \times m_\eta$ regression weights, and the elements of $\boldsymbol{\Gamma}$ are $m_\eta \times m_\xi$ regression weights. We assume that $\boldsymbol{\zeta}$ is uncorrelated with $\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}$ is uncorrelated with $\boldsymbol{\eta}$, and that $\boldsymbol{\delta}$ is uncorrelated with $\boldsymbol{\xi}$. We also assume that the means of $\boldsymbol{\varepsilon}$, and $\boldsymbol{\delta}$, and $\boldsymbol{\zeta}$ are zero, but it is not assumed that the means of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are zero. If the mean of $\boldsymbol{\xi}$ is denoted by $\boldsymbol{\kappa}$, the mean of $\boldsymbol{\eta}$ follows as

$$\boldsymbol{\mu}_\eta = (\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa})$$

The mean vectors of the observed indicators are given by

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_y \\ \boldsymbol{\mu}_x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y (\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa}) \\ \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x \boldsymbol{\kappa} \end{bmatrix}$$

The covariance matrix of the observed indicators follows as

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Lambda}_y\left((\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Gamma\Phi\Gamma}'(\mathbf{I}-\mathbf{B})^{-1\prime}+\mathbf{\Psi}^*\right)\mathbf{\Lambda}_y'+\mathbf{\Theta}_\varepsilon & \mathbf{\Lambda}_y(\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Gamma\Phi\Lambda}_x'+\mathbf{\Theta}_{\varepsilon\delta} \\ \mathbf{\Lambda}_x\mathbf{\Phi\Gamma}'\left(\mathbf{I}-\mathbf{B}\right)^{-1\prime}\mathbf{\Lambda}_y'+\mathbf{\Theta}_{\delta\varepsilon} & \mathbf{\Lambda}_x\mathbf{\Phi\Lambda}_x'+\mathbf{\Theta}_\delta \end{bmatrix}$$

where

$$\mathbf{\Psi}^* = (\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I}-\mathbf{B})^{-1\prime}$$

and $\mathbf{\Phi}$ denotes the covariance matrix of $\boldsymbol{\xi}$, $\mathbf{\Theta}_\varepsilon$ denotes the covariance matrix of $\boldsymbol{\varepsilon}$, $\mathbf{\Theta}_\delta$ denotes the covariance matrix of $\boldsymbol{\delta}$, $\mathbf{\Theta}_{\delta\varepsilon} = \mathbf{\Theta}_{\varepsilon\delta}'$ denotes the covariance matrix between $\boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$, and $\mathbf{\Psi}$ denotes the covariance matrix of $\boldsymbol{\zeta}$.

The q unknown parameters $\boldsymbol{\theta}$ of the extended LISREL model consist of the unknown elements of $\mathbf{\Lambda}_y$, $\mathbf{\Lambda}_x$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{\Theta}_\varepsilon$, $\mathbf{\Theta}_\delta$, $\mathbf{\Theta}_{\delta\varepsilon} = \mathbf{\Theta}_{\varepsilon\delta}'$, $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$.

Unstandardized solution

Suppose that $\hat{\boldsymbol{\theta}}$ denote the unstandardized estimators of the parameters $\boldsymbol{\theta}$ of the extended LISREL model as such that asymptotically

$$\hat{\boldsymbol{\theta}} \sim N\left(\boldsymbol{\theta}, \mathbf{H}\left(\boldsymbol{\theta}\right)\right)$$

The elements of $\boldsymbol{\theta}$ consists of intercepts, measurement weights, regression weights, variances, and covariances. The intercepts, measurement weights, regression weights, and covariances are unbounded parameters. As a result, the $100(1-\alpha)\%$ approximate confidence interval estimates of these parameters (Browne 1982) are given by

$$\left(\hat{\theta}_i - z_{\alpha/2}s\left(\hat{\theta}_i\right); \hat{\theta}_i + z_{\alpha/2}s\left(\hat{\theta}_i\right)\right)$$

where $\hat{\theta}_i$ denotes the estimate of θ_i, $z_{\alpha/2}$ denotes the $100(1-\alpha/2)\%$ critical value of the standard normal distribution and $s\left(\hat{\theta}_i\right) = \sqrt{\left[\mathbf{H}\left(\hat{\boldsymbol{\theta}}\right)\right]_{ii}}$ denotes the estimate of the standard error of the estimator of θ_i. If θ_i denotes a variance, then θ_i is a bounded parameter as such that $0 < \theta_i < \infty$. In this case, a logarithmic transformation is used as such that the $100(1-\alpha)\%$ approximate confidence interval estimate of θ_i (Browne 1982) follows as

$$\left(\hat{\theta}_i \exp\left(-z_{\alpha/2}s\left(\hat{\theta}_i\right)/\hat{\theta}_i\right); \hat{\theta}_i \exp\left(z_{\alpha/2}s\left(\hat{\theta}_i\right)/\hat{\theta}_i\right)\right)$$

1. Standardized solution

Let $\hat{\boldsymbol{\theta}}^*$ denote the standardized estimators of the parameters $\boldsymbol{\theta}$ of the extended LISREL model. By using the Delta method (Bishop, Feinberg, and Holland 1988), it follows that asymptotically

$$\hat{\boldsymbol{\theta}}^* \sim N\left(\boldsymbol{\theta}^*, \boldsymbol{\Delta}\mathbf{H}\left(\boldsymbol{\theta}\right)\boldsymbol{\Delta}'\right)$$

where $\boldsymbol{\Delta}$ denotes is Jacobian matrix of $\boldsymbol{\theta}^*$ with respect to $\boldsymbol{\theta}$. The elements of $\boldsymbol{\theta}^*$ consists of intercepts of the observed indicators, measurement weights, standardized regression weights, variances, standardized variances, covariances, and correlations. The intercepts of the observed variables, measurement weights, and covariances are unbounded parameters. As a result, the $100(1-\alpha)\%$ approximate confidence interval estimates of these parameters (Browne 1982) are given by

$$\left(\hat{\theta}_i^* - z_{\alpha/2}s\left(\hat{\theta}_i^*\right); \hat{\theta}_i^* + z_{\alpha/2}s\left(\hat{\theta}_i^*\right)\right)$$

where $\hat{\theta}_i^*$ denotes the estimate of θ_i^*, $z_{\alpha/2}$ denotes the $100(1-\alpha/2)\%$ critical value of the standard normal distribution and $s\left(\hat{\theta}_i^*\right) = \sqrt{\left[\boldsymbol{\Delta}\left(\hat{\boldsymbol{\theta}}^*, \hat{\boldsymbol{\theta}}\right)\mathbf{H}\left(\hat{\boldsymbol{\theta}}\right)\boldsymbol{\Delta}\left(\hat{\boldsymbol{\theta}}^*, \hat{\boldsymbol{\theta}}\right)'\right]_{ii}}$ denotes the estimate of the standard error of the estimator of θ_i^*. If θ_i^* denotes a variance, then θ_i^* is a bounded parameter as such that $0 < \theta_i^* < \infty$. In this case, a logarithmic transformation is used as such that the $100(1-\alpha)\%$ approximate confidence interval estimate of θ_i^* (Browne 1982) follows as

$$\left(\hat{\theta}_i \exp\left(-z_{\alpha/2}s\left(\hat{\theta}_i\right)\Big/\hat{\theta}_i\right); \hat{\theta}_i \exp\left(z_{\alpha/2}s\left(\hat{\theta}_i\right)\Big/\hat{\theta}_i\right)\right)$$

If θ_i^* denotes a standardized variance, then θ_i^* is a bounded parameter as such that $0 < \theta_i^* < 1$. The $100(1-\alpha)\%$ approximate confidence interval estimate of θ_i^* (Browne 1982) may be expressed as

$$\left(1\Bigg/\left(1 + \frac{1}{\left(\hat{\theta}_i^* - 1\right)}\exp\left(\frac{z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\hat{\theta}_i^*\left(1-\hat{\theta}_i^*\right)}\right)\right); 1\Bigg/\left(1 + \frac{1}{\left(\hat{\theta}_i^* - 1\right)}\exp\left(\frac{-z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\hat{\theta}_i^*\left(1-\hat{\theta}_i^*\right)}\right)\right)\right)$$

When θ_i^* is a standardized regression weight or a correlation, θ_i^* is bounded as such that $-1 < \theta_i^* < 1$. In this case, the Fisher z-transformation is used as such that the $100(1-\alpha)\%$ approximate confidence interval estimate of θ_i^* (Browne 1982) is given by

$$\left(\frac{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{-2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)-1}{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{-2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)+1};\frac{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)-1}{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)+1}\right)$$

2. Completely standardized solution

If $\hat{\boldsymbol{\theta}}^*$ denote the completely standardized estimators of the parameters $\boldsymbol{\theta}$ of the extended LISREL model, the asymptotic distribution of $\hat{\boldsymbol{\theta}}^*$ is obtained by means of the Delta method (Bishop, Feinberg, and Holland 1988) as

$$\hat{\boldsymbol{\theta}}^* \sim N\left(\boldsymbol{\theta}^*, \boldsymbol{\Delta}\mathbf{H}\left(\boldsymbol{\theta}\right)\boldsymbol{\Delta}'\right)$$

where $\boldsymbol{\Delta}$ denotes is Jacobian matrix of $\boldsymbol{\theta}^*$ with respect to $\boldsymbol{\theta}$. The elements of $\boldsymbol{\theta}^*$ consists of standardized measurement weights, standardized regression weights, standardized variances, and correlations.

If θ_i^* denotes a standardized variance, then θ_i^* is a bounded parameter as such that $0<\theta_i^*<1$. The $100(1-\alpha)\%$ approximate confidence interval estimate of θ_i^* (Browne 1982) may be expressed as

$$\left(1\Bigg/\left(1+\frac{1}{\left(\hat{\theta}_i^*-1\right)}\exp\left(\frac{z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\hat{\theta}_i^*\left(1-\hat{\theta}_i^*\right)}\right)\right);1\Bigg/\left(1+\frac{1}{\left(\hat{\theta}_i^*-1\right)}\exp\left(\frac{-z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\hat{\theta}_i^*\left(1-\hat{\theta}_i^*\right)}\right)\right)\right)$$

where $\hat{\theta}_i^*$ denotes the estimate of θ_i^*, $z_{\alpha/2}$ denotes the $100(1-\alpha/2)\%$ critical value of the standard normal distribution and $s\left(\hat{\theta}_i^*\right)=\sqrt{\left[\boldsymbol{\Delta}\left(\hat{\boldsymbol{\theta}}^*,\hat{\boldsymbol{\theta}}\right)\mathbf{H}\left(\hat{\boldsymbol{\theta}}\right)\boldsymbol{\Delta}\left(\hat{\boldsymbol{\theta}}^*,\hat{\boldsymbol{\theta}}\right)'\right]_{ii}}$ denotes the estimate of the standard error of the estimator of θ_i^*.

When θ_i^* is a standardized measurement weight, a standardized regression weight, or a correlation, θ_i^* is bounded as such that $-1<\theta_i^*<1$. In this case, the Fisher *z*-transformation is used a such that the $100(1-\alpha)\%$ approximate confidence interval estimate of θ_i^* (Browne 1982) is given by

$$\left(\frac{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{-2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)-1}{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{-2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)+1};\frac{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)-1}{\left(\frac{1+\hat{\theta}_i^*}{1-\hat{\theta}_i^*}\right)\exp\left(\frac{2z_{\alpha/2}s\left(\hat{\theta}_i^*\right)}{\left(1-\hat{\theta}_i^{*2}\right)}\right)+1}\right)$$

References

Bishop, Y.M.M., Feinberg, S.E. & Holland, P.W. (1988). *Discrete Multivariate Analysis: Theory and Practice*. Cambridge: MIT Press.

Browne, M.W. (1982). Covariance Structures. In D.M. Hawkins (Ed.), *Topics in Applied Multivariate Analysis*, pp. 72-141. Cambridge: Cambridge University Press.

Jöreskog, K.G. & Sörbom, D. (1999). *LISREL 8: User's reference guide*. Lincolnwood, IL: Scientific Software International, Inc.