Standard error estimates for standardized solutions

The LISREL model for observed and latent variables

The LISREL model (Jöreskog 1973, 1977) for population covariance matrices may be expressed as

$$\mathbf{y} = \mathbf{\Lambda}_y \boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

$$\mathbf{x} = \mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}$$

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \mathbf{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}$$

where $\mathbf{y}$ and $\mathbf{x}$ denote p_η and p_ξ indicators of the m_η endogenous latent variables, $\boldsymbol{\eta}$, and the m_ξ exogenous latent variables, $\boldsymbol{\xi}$, respectively, $\mathbf{\Lambda}_y$ and $\mathbf{\Lambda}_x$ are $p_y \times m_\eta$ and $p_x \times m_\xi$ matrices of factor loadings, respectively, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ denote p_η and p_ξ measurement errors, respectively, $\mathbf{B}$ and $\mathbf{\Gamma}$ are $m_\eta \times m_\eta$ and $m_\eta \times m_\xi$ matrices of regression weights, respectively, and the elements of $\boldsymbol{\zeta}$ denote m_η error variables.

The $t \times 1$ vector, $\mathbf{z}$, consisting of all the variables of the LISREL model follows as

$$\mathbf{z} = \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \\ \boldsymbol{\eta} \\ \boldsymbol{\xi} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \\ \boldsymbol{\zeta} \end{pmatrix}$$

The model for the relationships between all the variables of the LISREL model may then be expressed as

$$\mathbf{z} = \mathbf{B}_t \mathbf{z} + \mathbf{z}_e$$

where

$$\mathbf{B}_t = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_y & \mathbf{0} & \mathbf{I}_{p_y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_x & \mathbf{0} & \mathbf{I}_{p_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \boldsymbol{\Gamma} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{m_\eta} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix and

$$\mathbf{z}_e = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\xi} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \\ \boldsymbol{\zeta} \end{pmatrix}$$

The covariance matrix, $\boldsymbol{\Phi}_t$, of $\mathbf{z}_e$ follows as

$$\boldsymbol{\Phi}_t = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Phi} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Theta}_\varepsilon & \boldsymbol{\Theta}_{\varepsilon\delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Theta}_{\delta\varepsilon} & \boldsymbol{\Theta}_\delta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Psi} \end{pmatrix}$$

where $\boldsymbol{\Phi}$, $\boldsymbol{\Theta}_\varepsilon$, $\boldsymbol{\Theta}_\delta$, and $\boldsymbol{\Psi}$ denote the covariance matrices of $\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\delta}$, and $\boldsymbol{\zeta}$, respectively and $\boldsymbol{\Theta}_{\varepsilon\delta} = \boldsymbol{\Theta}'_{\delta\varepsilon}$ denotes the covariance matrix between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$. The $t \times t$ covariance matrix of $\mathbf{z}$, Υ_t, may then be expressed as

$$\Upsilon_t = \left(\mathbf{I}_t - \mathbf{B}_t\right)^{-1} \boldsymbol{\Phi}_t \left(\mathbf{I}_t - \mathbf{B}_t\right)^{-1\prime} = \boldsymbol{\Lambda}_t \boldsymbol{\Phi}_t \boldsymbol{\Lambda}'_t$$

where $\boldsymbol{\Lambda}_t = \left(\mathbf{I}_t - \mathbf{B}_t\right)^{-1}$. The $(p+m)\times(p+m)$ covariance matrix, Υ, of the $p = p_y + p_x$ observed variables and $m = m_\eta + m_\xi$ latent variables, follows as

$$\Upsilon = \begin{bmatrix} \mathbf{I}_{p+m} & \mathbf{0} \end{bmatrix} \left(\mathbf{I}_t - \mathbf{B}_t\right)^{-1} \boldsymbol{\Phi}_t \left(\mathbf{I}_t - \mathbf{B}_t\right)^{-1\prime} \begin{bmatrix} \mathbf{I}_{p+m} & \mathbf{0} \end{bmatrix}' = \boldsymbol{\Lambda} \boldsymbol{\Phi}_t \boldsymbol{\Lambda}'$$

where $\mathbf{\Lambda}$ denotes the $(p+m)\times t$ matrix consisting of the first $p+m$ rows of $\mathbf{\Lambda}_t$. In terms of the parameter matrices of the LISREL model, we obtain that

$$\Upsilon = \begin{pmatrix} \mathbf{\Sigma}_{yy} & \mathbf{\Sigma}_{yx} & \mathbf{\Sigma}_{y\eta} & \mathbf{\Sigma}_{y\xi} \\ \mathbf{\Sigma}_{xy} & \mathbf{\Sigma}_{xx} & \mathbf{\Sigma}_{x\eta} & \mathbf{\Sigma}_{x\xi} \\ \mathbf{\Sigma}_{\eta y} & \mathbf{\Sigma}_{\eta x} & \mathbf{\Sigma}_{\eta\eta} & \mathbf{\Sigma}_{\eta\xi} \\ \mathbf{\Sigma}_{\xi y} & \mathbf{\Sigma}_{\xi x} & \mathbf{\Sigma}_{\xi\eta} & \mathbf{\Sigma}_{\xi\xi} \end{pmatrix}$$

where

$$\mathbf{\Sigma}_{yy} = \mathbf{\Lambda}_y \left((\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Gamma\Phi\Gamma}'(\mathbf{I}-\mathbf{B})^{-1\prime} + \mathbf{\Psi}^* \right) \mathbf{\Lambda}'_y + \mathbf{\Theta}_\varepsilon$$

$$\mathbf{\Sigma}_{xy} = \mathbf{\Sigma}'_{yx} = \mathbf{\Lambda}_x \mathbf{\Phi\Gamma}' \left(\mathbf{I}-\mathbf{B}\right)^{-1\prime} \mathbf{\Lambda}'_y + \mathbf{\Theta}_{\delta\varepsilon}$$

$$\mathbf{\Sigma}_{xx} = \mathbf{\Lambda}_x \mathbf{\Phi\Lambda}'_x + \mathbf{\Theta}_\delta$$

$$\mathbf{\Sigma}_{\eta y} = \left((\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Gamma\Phi\Gamma}'(\mathbf{I}-\mathbf{B})^{-1\prime} + \mathbf{\Psi}^* \right) \mathbf{\Lambda}'_y$$

$$\mathbf{\Sigma}_{\eta x} = \mathbf{\Sigma}'_{\eta x} = \left(\mathbf{I}-\mathbf{B}\right)^{-1} \mathbf{\Gamma\Phi\Lambda}'_x$$

$$\mathbf{\Sigma}_{\eta\eta} = (\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Gamma\Phi\Gamma}'(\mathbf{I}-\mathbf{B})^{-1\prime} + \mathbf{\Psi}^*$$

$$\mathbf{\Sigma}_{\xi y} = \mathbf{\Phi\Gamma}'(\mathbf{I}-\mathbf{B})^{-1\prime} \mathbf{\Lambda}'_y$$

$$\mathbf{\Sigma}_{\xi x} = \mathbf{\Sigma}'_{x\xi} = \mathbf{\Phi\Lambda}'_x$$

$$\mathbf{\Sigma}_{\xi\eta} = \mathbf{\Sigma}'_{\eta\xi} = \mathbf{\Phi\Gamma}'(\mathbf{I}-\mathbf{B})^{-1\prime}$$

$$\mathbf{\Sigma}_{\xi\xi} = \mathbf{\Phi}_\xi$$

where

$$\mathbf{\Psi}^* = (\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I}-\mathbf{B})^{-1\prime}$$

Completely standardized solution

Suppose now that $\hat{\mathbf{B}}_t$ and $\hat{\mathbf{\Phi}}_t$ denote the unstandardized estimators of $\mathbf{B}_t$ and $\mathbf{\Phi}_t$, respectively. The reproduced covariance matrix of the observed and latent variables may then be expressed as

$$\hat{\Upsilon} = \hat{\mathbf{\Lambda}}\hat{\mathbf{\Phi}}_t\hat{\mathbf{\Lambda}}'$$

The completely standardized covariance matrix of the observed and latent variables follows as

$$\hat{\Upsilon}^* = \mathbf{D}_{\hat{\Upsilon}}^{-1/2}\hat{\mathbf{\Lambda}}\hat{\mathbf{\Phi}}_t\hat{\mathbf{\Lambda}}'\mathbf{D}_{\hat{\Upsilon}}^{-1/2}$$

where $\mathbf{D}_{\hat{\Upsilon}}^{-1/2}$ denotes a $(p+m)\times(p+m)$ diagonal matrix with the reciprocals of the estimated standard deviations of the observed and latent variables on the diagonal.

The relationships between the unstandardized and the completely standardized estimators are given by

$$\hat{\mathbf{B}}_t^* = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_y^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_x^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta^{-1} & \hat{\mathbf{D}}_\eta^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \hat{\mathbf{B}}_t \begin{pmatrix} \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\xi & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta & \hat{\mathbf{D}}_\xi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$\hat{\mathbf{\Phi}}_t^* = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\xi^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_y^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_x^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta^{-1} \end{pmatrix} \hat{\mathbf{\Phi}}_t \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\xi^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_y^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_x^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta^{-1} \end{pmatrix}$$

where $\hat{\mathbf{D}}_y$, $\hat{\mathbf{D}}_x$, $\hat{\mathbf{D}}_\eta$, and $\hat{\mathbf{D}}_\xi$ denote diagonal matrices with the estimated standard deviations of the elements of $\mathbf{y}$, $\mathbf{x}$, $\boldsymbol{\eta}$, and $\boldsymbol{\xi}$ on the diagonal, respectively.

Suppose that the vector $\boldsymbol{\theta}$ consists of the q unknown elements of $\mathbf{B}$ and $\mathbf{\Phi}$. Let $\hat{\boldsymbol{\theta}}$ denotes the unstandardized estimator of $\boldsymbol{\theta}$ as such that asymptotically

$$\hat{\boldsymbol{\theta}} \sim N(\boldsymbol{\theta}, \mathbf{H}(\boldsymbol{\theta}))$$

By using the Delta method (Bishop, Fienberg, and Holland 1988), the asymptotic distribution of the completely standardized estimator, $\hat{\boldsymbol{\theta}}^*$, of $\boldsymbol{\theta}$ follows as

$$\hat{\boldsymbol{\theta}}^* \sim N\left(\boldsymbol{\theta}^*, \boldsymbol{\Delta}\mathbf{H}(\boldsymbol{\theta})\boldsymbol{\Delta}'\right)$$

where $\boldsymbol{\Delta} = \dfrac{\delta\boldsymbol{\theta}^*}{\delta\boldsymbol{\theta}'}$ and the elements of $\boldsymbol{\theta}^*$ are the unknown elements of $\mathbf{B}^*$ and $\boldsymbol{\Phi}^*$. Typical elements of $\mathbf{B}^*$ and $\boldsymbol{\Phi}^*$ are given by

$$\beta_{ij}^* = \upsilon_{ii}^{-1/2}\upsilon_{jj}^{1/2}\beta_{ij}$$

and

$$\phi_{ij}^* = \upsilon_{rr}^{-1/2}\upsilon_{ss}^{-1/2}\phi_{ij}$$

respectively where r and s are defined as

$$r = \begin{cases} i & \text{if } i \le p+m \\ i-p-m & \text{if } i > p+m \end{cases} \text{ and } s = \begin{cases} j & \text{if } j \le p+m \\ j-p-m & \text{if } j > p+m \end{cases}$$

respectively. Suppose that the sets $I_{\mathbf{B}}$ and $I_{\boldsymbol{\Phi}}$ are sets containing the row and column positions of the unknown elements of $\mathbf{B}$ and $\boldsymbol{\Phi}$, respectively, i.e.

$$I_{\mathbf{B}} = \left\{(i,j) : \beta_{ij} \in \boldsymbol{\theta}\right\} \text{ and } I_{\boldsymbol{\Phi}} = \left\{(i,j) : \phi_{ij} \in \boldsymbol{\theta}\right\}$$

respectively. The partial derivatives of the diagonal elements of Υ with respect to the elements of $\mathbf{B}$ and $\boldsymbol{\Phi}$ may then be expressed as

$$\frac{\delta\upsilon_{ii}}{\delta\beta_{kl}} = \begin{cases} 2\lambda_{ik}\upsilon_{il} & \text{if } (k,l) \in I_{\mathbf{B}} \\ 0 & \text{if } (k,l) \notin I_{\mathbf{B}} \end{cases}$$

and

$$\frac{\delta\upsilon_{ii}}{\delta\phi_{kl}} = \begin{cases} 2\left(1+\delta_{kl}\right)^{-1}\lambda_{ik}\lambda_{il} & \text{if } (k,l) \in I_{\boldsymbol{\Phi}} \\ 0 & \text{if } (k,l) \notin I_{\boldsymbol{\Phi}} \end{cases}$$

respectively where δ_{kl} denotes the Kronecker delta, i.e.

$$\delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \ne l \end{cases}$$

Typical elements of $\boldsymbol{\Delta}$ may be expressed as

$$[\boldsymbol{\Delta}]_{ij,kl} = \begin{cases} \dfrac{\delta\beta_{ij}^*}{\delta\beta_{kl}} & \text{if } (i,j)\in I_{\mathbf{B}} \text{ and } (k,l)\in I_{\mathbf{B}} \\ \dfrac{\delta\beta_{ij}^*}{\delta\phi_{kl}} & \text{if } (i,j)\in I_{\mathbf{B}} \text{ and } (k,l)\in I_{\boldsymbol{\Phi}} \\ \dfrac{\delta\phi_{ij}^*}{\delta\beta_{kl}} & \text{if } (i,j)\in I_{\boldsymbol{\Phi}} \text{ and } (k,l)\in I_{\mathbf{B}} \\ \dfrac{\delta\phi_{ij}^*}{\delta\phi_{kl}} & \text{if } (i,j)\in I_{\boldsymbol{\Phi}} \text{ and } (k,l)\in I_{\boldsymbol{\Phi}} \end{cases}$$

where

$$\frac{\delta\beta_{ij}^*}{\delta\beta_{kl}} = \lambda_{jk}\upsilon_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-1/2}\beta_{ij} - \lambda_{ik}\upsilon_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{1/2}\beta_{ij} + \upsilon_{ii}^{-1/2}\upsilon_{jj}^{1/2}\frac{\delta\beta_{ij}}{\delta\beta_{kl}}$$

$$\frac{\delta\beta_{ij}^*}{\delta\phi_{kl}} = \left[\lambda_{jk}\lambda_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-1/2} - \lambda_{ik}\lambda_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{1/2}\right](1+\delta_{kl})^{-1}\beta_{ij}$$

$$\frac{\delta\phi_{ij}^*}{\delta\beta_{kl}} = -\left[\lambda_{ik}\upsilon_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{-1/2} + \lambda_{jk}\upsilon_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-3/2}\right]\phi_{ij}$$

$$\frac{\delta\phi_{ij}^*}{\delta\phi_{kl}} = -\left[\lambda_{ik}\lambda_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{-1/2} + \lambda_{jk}\lambda_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-3/2}\right]\left(1+\delta_{kl}\right)^{-1}\phi_{ij} + \upsilon_{ii}^{-1/2}\upsilon_{jj}^{-1/2}\frac{\delta\phi_{ij}}{\delta\phi_{kl}}$$

The standard error estimates of the completely standardized estimators of the elements of $\boldsymbol{\theta}$ are obtained as the positive square roots of the diagonal elements of the estimated asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}^*$ which is given by

$$\mathbf{H}^*\left(\hat{\boldsymbol{\theta}}^*\right) = \boldsymbol{\Delta}\left(\hat{\boldsymbol{\theta}}^*,\hat{\boldsymbol{\theta}}\right)\mathbf{H}\left(\hat{\boldsymbol{\theta}}\right)\boldsymbol{\Delta}\left(\hat{\boldsymbol{\theta}}^*,\hat{\boldsymbol{\theta}}\right)'$$

These standard error estimates are numerically equivalent to those obtained by transforming correlation structures to covariance structures and fitting the transformed covariance structures correctly to the sample correlation matrices by using the theory and methods for covariance structures proposed by Shapiro and Browne (1990) which are implemented in Steiger (1995) and Browne and Mels (1996). Whenever a LISREL model without parameter equality constraints is fitted to a sample correlation matrix, the standard error estimates of the completely standardized solution are the correct standard error estimates which addresses the issue of incorrect standard error estimates for correlation matrices pointed out by Cudeck (1989).

Standardized solution

Let $\hat{\mathbf{B}}_t$ and $\hat{\mathbf{\Phi}}_t$ denote the unstandardized estimators of $\mathbf{B}_t$ and $\mathbf{\Phi}_t$, respectively and let $\hat{\mathbf{B}}_t^*$ and $\hat{\mathbf{\Phi}}_t^*$ denote the corresponding standardized estimators. The relationships between the unstandardized and the standardized estimators of $\mathbf{B}_t$ and $\mathbf{\Phi}_t$ are given by

$$\hat{\mathbf{B}}_t^* = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_y} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_x} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta^{-1} & \hat{\mathbf{D}}_\eta^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \hat{\mathbf{B}}_t \begin{pmatrix} \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\xi & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta & \hat{\mathbf{D}}_\xi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$\hat{\mathbf{\Phi}}_t^* = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\xi^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta^{-1} \end{pmatrix} \hat{\mathbf{\Phi}}_t \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\xi^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_\eta^{-1} \end{pmatrix}$$

respectively where $\hat{\mathbf{D}}_\eta$ and $\hat{\mathbf{D}}_\xi$ denote diagonal matrices with the estimated standard deviations of the elements of $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ on the diagonal, respectively.

Suppose that the vector $\boldsymbol{\theta}$ consists of the q unknown elements of $\mathbf{B}$ and $\mathbf{\Phi}$. Let $\hat{\boldsymbol{\theta}}$ denotes the unstandardized estimator of $\boldsymbol{\theta}$ as such that asymptotically

$$\hat{\boldsymbol{\theta}} \sim N\left(\boldsymbol{\theta}, \mathbf{H}(\boldsymbol{\theta})\right)$$

By using the Delta method (Bishop, Fienberg, and Holland 1988), the asymptotic distribution of the standardized estimator, $\hat{\boldsymbol{\theta}}^*$, of $\boldsymbol{\theta}$ follows as

$$\hat{\boldsymbol{\theta}}^* \sim N\left(\boldsymbol{\theta}^*, \mathbf{\Delta}\mathbf{H}(\boldsymbol{\theta})\mathbf{\Delta}'\right)$$

where $\mathbf{\Delta} = \dfrac{\delta\boldsymbol{\theta}^*}{\delta\boldsymbol{\theta}'}$ and the elements of $\boldsymbol{\theta}^*$ are the unknown elements of $\mathbf{B}^*$ and $\mathbf{\Phi}^*$. Typical elements of $\mathbf{B}^*$ and $\mathbf{\Phi}^*$ are given by

$$\beta_{ij}^{*} = \begin{cases} \upsilon_{jj}^{1/2}\beta_{ij} & \text{if } i \le p \\ \upsilon_{ii}^{-1/2}\upsilon_{jj}^{1/2}\beta_{ij} & \text{if } i > p \end{cases}$$

and

$$\phi_{ij}^{*} = \begin{cases} \upsilon_{rr}^{-1/2}\upsilon_{ss}^{-1/2}\phi_{ij} & \text{if } (i,j) \in I_1 \\ \phi_{ij} & \text{if } (i,j) \in I_2 \end{cases}$$

respectively where r and s are defined as

$$r = \begin{cases} i & \text{if } i \le p+m \\ i-p-m & \text{if } i > p+m \end{cases} \text{ and } s = \begin{cases} j & \text{if } j \le p+m \\ j-p-m & \text{if } j > p+m \end{cases}$$

respectively and where the sets I_1 and I_2 are defined as

$$I_1 = \{(i,j) : i, j \le p+m \text{ or } i, j > 2p+m\} \text{ and } I_2 = \{(i,j) : p+m < i, j \le 2p+m\}$$

respectively.

Suppose that the sets $I_{\mathbf{B}}$ and $I_{\mathbf{\Phi}}$ are sets containing the row and column positions of the unknown elements of $\mathbf{B}$ and $\mathbf{\Phi}$, respectively, i.e.

$$I_{\mathbf{B}} = \{(i,j) : \beta_{ij} \in \boldsymbol{\theta}\} \text{ and } I_{\mathbf{\Phi}} = \{(i,j) : \phi_{ij} \in \boldsymbol{\theta}\}$$

respectively. The partial derivatives of the diagonal elements of Υ with respect to the elements of $\mathbf{B}$ and $\mathbf{\Phi}$ may then be expressed as

$$\frac{\delta\upsilon_{ii}}{\delta\beta_{kl}} = \begin{cases} 2\lambda_{ik}\upsilon_{il} & \text{if } (k,l) \in I_{\mathbf{B}} \\ 0 & \text{if } (k,l) \notin I_{\mathbf{B}} \end{cases}$$

and

$$\frac{\delta\upsilon_{ii}}{\delta\phi_{kl}} = \begin{cases} 2(1+\delta_{kl})^{-1}\lambda_{ik}\lambda_{il} & \text{if } (k,l) \in I_{\mathbf{\Phi}} \\ 0 & \text{if } (k,l) \notin I_{\mathbf{\Phi}} \end{cases}$$

respectively where δ_{kl} denotes the Kronecker delta, i.e.

$$\delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \ne l \end{cases}$$

Typical elements of $\mathbf{\Delta}$ may be expressed as

$$[\mathbf{\Delta}]_{ij,kl} = \begin{cases} \dfrac{\delta\beta^*_{ij}}{\delta\beta_{kl}} & \text{if } (i,j) \in I_{\mathbf{B}} \text{ and } (k,l) \in I_{\mathbf{B}} \\ \dfrac{\delta\beta^*_{ij}}{\delta\phi_{kl}} & \text{if } (i,j) \in I_{\mathbf{B}} \text{ and } (k,l) \in I_{\mathbf{\Phi}} \\ \dfrac{\delta\phi^*_{ij}}{\delta\beta_{kl}} & \text{if } (i,j) \in I_{\mathbf{\Phi}} \text{ and } (k,l) \in I_{\mathbf{B}} \\ \dfrac{\delta\phi^*_{ij}}{\delta\phi_{kl}} & \text{if } (i,j) \in I_{\mathbf{\Phi}} \text{ and } (k,l) \in I_{\mathbf{\Phi}} \end{cases}$$

where

$$\frac{\delta\beta^*_{ij}}{\delta\beta_{kl}} = \begin{cases} \lambda_{jk}\upsilon_{jl}\upsilon_{jj}^{-1/2}\beta_{ij} + \upsilon_{jj}^{1/2}\dfrac{\delta\beta_{ij}}{\delta\beta_{kl}} & \text{if } i \le p \\ \lambda_{jk}\upsilon_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-1/2}\beta_{ij} - \lambda_{ik}\upsilon_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{1/2}\beta_{ij} + \upsilon_{ii}^{-1/2}\upsilon_{jj}^{1/2}\dfrac{\delta\beta_{ij}}{\delta\beta_{kl}} & \text{if } i > p \end{cases}$$

$$\frac{\delta\beta^*_{ij}}{\delta\phi_{kl}} = \begin{cases} (1+\delta_{kl})^{-1}\lambda_{jk}\lambda_{jl}\upsilon_{jj}^{-1/2}\beta_{ij} & \text{if } i \le p \\ \left[\lambda_{jk}\lambda_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-1/2} - \lambda_{ik}\lambda_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{1/2}\right](1+\delta_{kl})^{-1}\beta_{ij} & \text{if } i > p \end{cases}$$

$$\frac{\delta\phi^*_{ij}}{\delta\beta_{kl}} = \begin{cases} -\left[\lambda_{ik}\upsilon_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{-1/2} + \lambda_{jk}\upsilon_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-3/2}\right]\phi_{ij} & \text{if } (i,j) \in I_1 \\ 0 & \text{if } (i,j) \in I_2 \end{cases}$$

$$\frac{\delta\phi^*_{ij}}{\delta\phi_{kl}} = \begin{cases} -\left[\lambda_{ik}\lambda_{il}\upsilon_{ii}^{-3/2}\upsilon_{jj}^{-1/2} + \lambda_{jk}\lambda_{jl}\upsilon_{ii}^{-1/2}\upsilon_{jj}^{-3/2}\right](1+\delta_{kl})^{-1}\phi_{ij} + \upsilon_{ii}^{-1/2}\upsilon_{jj}^{-1/2}\dfrac{\delta\phi_{ij}}{\delta\phi_{kl}} & \text{if } (i,j) \in I_1 \\ \dfrac{\delta\phi_{ij}}{\delta\phi_{kl}} & \text{if } (i,j) \in I_2 \end{cases}$$

The standard error estimates of the completely standardized estimators of the elements of $\boldsymbol{\theta}$ are obtained as the positive square roots of the diagonal elements of the estimated asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}^*$ which is given by

$$\mathbf{H}^*\left(\hat{\boldsymbol{\theta}}^*\right) = \mathbf{\Delta}\left(\hat{\boldsymbol{\theta}}^*, \hat{\boldsymbol{\theta}}\right)\mathbf{H}\left(\hat{\boldsymbol{\theta}}\right)\mathbf{\Delta}\left(\hat{\boldsymbol{\theta}}^*, \hat{\boldsymbol{\theta}}\right)'$$

References

Bishop, Y.M.M., Feinberg, S.E. & Holland, P.W. (1988). *Discrete Multivariate Analysis: Theory and Practice*. Cambridge: MIT Press.

Browne, M.W. & Mels, G. (1996). *Path Analysis: RAMONA, SYSTAT FOR DOS; Advanced Applications, Version 6.* Evanston, IL: Systat Inc., 163-224.

Cudeck, R. (1989). The Analysis of Correlation Matrices using Covariance Structure Models. *Psychological Bulletin*, **105**, 317-327.

Jöreskog, K.G. (1973). A General Method for Estimating a Linear Structural Equation System. In: A.S. Goldberger & O.D. Duncan (Eds.), *Structural Equation Models in the Social Sciences*. New York: Seminar Press.

Jöreskog, K.G. (1977). Structural Equation Models in the Social Sciences: Specification, Estimation and Testing. In P.R. Krishnaiah (Ed.), *Applications of Statistics*, pp. 265-287. Amsterdam: North Holland.

Shapiro, A. & Browne, M.W. (1990). On the Treatment of Correlation Structures as Covariance Structures. *Linear Algebra and its Applications*, **127**, 547-567.

Steiger, J.H. (1995). Structural Equation Modelling. In: *Statistica for Windows (Volume III): Statistics II, 2nd Edition*, pp. 3539-3688. Tulsa: StatSoft, Inc.